Week 9

Reductions

Anakin

Outline

Review

Reductions

Updates!

Weekly updates:

- Stickers have been ordered
- I'll let everyone know when they get here

Section 1

Review

What We Know

- We say that M ***recognizes/accepts*** L if for any input $w \in L$, M accepts w.
  - If $w \in L$, then M must **accept** w
  - If $w \notin L$, then M can **reject** or even **never halt**
- M ***decides*** L if for any input w, M accepts if $w \in L$ and rejects otherwise.
  - If $w \in L$, then M must **accept** w
  - If $w \notin L$, then M must **reject** w
  - Either way, M must halt on all inputs

What We Know

- L is **recognizable** if there exists some TM M that recognizes it
- L is **decidable** if there exists some TM M that decides it
- Some languages are unrecognizable or undecidable

Key Examples

- $HALT_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \}$
  - We showed this language is undecidable
- $A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$
  - We showed this language is undecidable
- $\overline{A_{\text{TM}}}$ = complement of A_{TM}
  - We showed this language is unrecognizable

How Do We Figure Out More?

- The proofs for all of these were long and confusing
- What if we wanted to prove other languages are undecidable/unrecognizable?
  - Do you want to do all that work again?
  - I don't
- What if we could leverage our previous proofs?

Section 2

Reductions

Proof that π is irrational

If π is rational, then it can be expressed as a fraction $\frac{a}{b}$, where $(a, b \in \mathbb{Z}, b \neq 0)$.

However, this contradicts Johann Heinrich Lambert's 1761 proof that π is irrational.

Therefore, π must be irrational. $\square$

What Is A Reduction?

- As stupid as it is, the previous proof follows the idea of a reduction
- We assume something and reach a conclusion that contradicts a known result, thus proving our assumption was incorrect

What Is A Reduction?

- A **decidability** reduction is a proof of the following form:
  - Suppose language L is decidable, then there exists a TM M that decides it
  - We can use M as a blackbox in some other algorithm to decide your favorite undecidable problem
  - This contradicts the fact that your chosen problem is undecidable
  - Thus L is undecidable

Reductions = Algorithms

- This is the format of a **Turing Reduction**[1]
- You are given access to some blackbox oracle
  - Imagine the oracle like some library function you import
  - You know what it takes as input and gives as output
  - You just don't know exactly how it works
- You write an algorithm using this oracle
- If you are reducing from A to B, then your algorithm should accept **if and only if** the input is in language A

[1] CS 374 talks about mapping reductions, which are equivalent but more confusing

Difficulty

- Reductions play an extremely central in complexity theory since it allows to talk about how **hard problems** are
- If we can reduce Problem A to Problem B, solving A cannot be harder than solving B
  - Solving B gives a solution to A
- Thus if A reduces to B and we know something about A, we learn:
  - if A is undecidable, then B is undecidable
  - if A is unrecognizable, then B is unrecognizable

Decidability Reduction

$$E_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$$

- We will show this is undecidable using a reduction
- We know that the Halting Problem is undecidable, so let's use that
- Suppose that $E(\langle M \rangle)$ is a TM that decides E_{TM}
- We will design a TM $H(\langle M, w \rangle)$ that decides the Halting Problem

Decidability Reduction

Suppose that $E(\langle M \rangle)$ is a TM that decides E_{TM}

$\underline{\text{CreateM}_1(\langle M, w \rangle)}$:
  Create a TM M_1 that does the following:
    On input w, accept if $M(w)$ halts
    else, reject
  return $\langle M_1 \rangle$

$\underline{\text{H}(\langle M, w \rangle)}$:
  $\langle M_1 \rangle \leftarrow \text{CreateM}_1(\langle M, w \rangle)$
  run $E(\langle M_1 \rangle)$
  if E accepts, reject; if E rejects, accept

E accepts $\iff$ $L(M_1)$ is empty $\iff$ M does not halt on input w

Recognizability Reduction

Recall:

- $A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$
- A_{TM} is recognizable but not decidable
- So $\overline{A_{\text{TM}}}$ is not recognizable

We also have the following

- A reduces to B if and only if $\overline{A}$ reduces to $\overline{B}$
- If A reduces to B and B is recognizable, then A is recognizable
- If A reduces to B and A is unrecognizable, then B is unrecognizable

So to show that some problem B is unrecognizable, we can show that A_{TM} reduces to $\overline{B}$ since that's the same as showing $\overline{A_{\text{TM}}}$ reduces to B

Recognizability Reduction

$$EQ_{\text{TM}} = \{\, \langle M_1,\ M_2 \rangle \mid M_1,\ M_2 \text{ are TMs and } L(M_1) = L(M_2) \,\}$$

Using what we said before, we can show that EQ_{TM} is not recognizable by showing that A_{TM} reduces to $\overline{EQ_{\text{TM}}}$

Recognizability Reduction

Suppose that $\overline{EQ}(\langle M_1, M_2 \rangle)$ is a TM that recognizes

$$\overline{EQ_{\text{TM}}} = \{ \langle M_1,\ M_2 \rangle \mid M_1,\ M_2 \text{ are TMs and } L(M_1) \neq L(M_2) \}$$

We will reduce from A_{TM} as follows. Given M and w we will construct the following two machines

$\text{M}_1(\langle M, w \rangle)$:
on any input:
reject

$\text{M}_2(\langle M, w \rangle)$:
on any input:
return $M(w)$

Recognizability Reduction

We build a machine A that recognizes A_{TM}

```
M_1(w):
    on any input:
        reject
```

```
M_2(w):
    on any input:
        return M(w)
```

```
A(⟨M, w⟩):
    Create M_1, M_2 as described
    return EQ̄(⟨M_1, M_2⟩)
```

M accepts $w \iff \overline{EQ}(\langle M_1, M_2 \rangle)$ accepts $\iff L(M_1) \neq L(M_2)$

Questions?

Questions!

1: Recall L is **co-recognizable** if there exists some TM M that recognizes it's complement $\Sigma^* \setminus L$
Modify the previous proof and reduce A_{TM} to $EQ(\langle M_1, M_2 \rangle)$ to show that $EQ(\langle M_1, M_2 \rangle)$ is not co-recognizable)

2: Show that if some language A is recognizable, and A reduces to $\overline{A}$, then A is decidable

So long and thanks for all the fish!
— DOUGLAS ADAMS (1979)