Introduction to Qubits

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MATRIX

- We've got two events this week - free pizza and snacks at both!
- On the ruler and compass problem: tomorrow, Oct 3rd, 6:30pm in 1051 Lincoln Hall
- Trivia Night: Thursday, Oct 5th, 6:30pm in 1022 Lincoln Hall
- Scan the QR code for our Discord

Outline

What is a qubit?

Measuring and representing qubit states

Qubit errors and why it's so hard to correct them

Fixing errors (?)

Important notes

- This is part 1 of 2 in a series on quantum error correction. Make sure to come to the next one!
- Some advice: when learning quantum mechanics, treat it like a game with rules to play. (Unless you're a physicist.)

Section 1

What is a qubit?

Classical versus Quantum Bits

- A classical bit can be either 0 or 1
- Quantum bits (qubits) can be 0, 1, or a superposition of the two
- What does that mean *mathematically*?

Qubits, Formally

- A qubit is represented by a statevector in a state space
- The state space is a *Hilbert space*, aka $\mathbb{C}^k$
  - There are other Hilbert spaces, but we only really need $\mathbb{C}^k$
  - Hilbert spaces have inner products and associated norms that make it a complete metric space: these are linear algebra concepts you don't need to understand, but the point is having these makes everything nicer
- A statevector is simply a unit (norm 1) vector of the state space
- Examples: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

What was that notation!?

- Quantum mechanics uses bra-ket notation
- $|\phi\rangle$ (a "ket") is a vector, $\langle\phi|$ (a "bra") is its conjugate transpose
  - Example: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$ or $|\chi\rangle = \begin{pmatrix} 0 \\ i \end{pmatrix}, \langle\chi| = \begin{pmatrix} 0 & -i \end{pmatrix}$
- Putting the two together: $\langle\phi|\psi\rangle$ is the inner product of $|\phi\rangle$ and $|\psi\rangle$
- Examples:
  - $\langle\chi|0\rangle = (1 * 0) + (0 * i) = 0$
  - $\langle\chi|\chi\rangle = (0 * 0) + (-i * i) = 1$

More notation

I promise it'll be over soon

- $|0\rangle$ and $|1\rangle$ are an orthonormal basis over $\mathbb{C}^2$; by convention, we denote this the computational basis
  - We could use any orthonormal basis! There's nothing special about $\{|0\rangle, |1\rangle\}$ versus the other ONBs except that it's the easiest to write down
- An arbitrary statevector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
- Recall we said statevectors have unit length; in other words, $\langle\psi|\psi\rangle = 1$
- Exercise: $\langle\psi|\psi\rangle = 1$ enforces conditions on α and β. What are they?

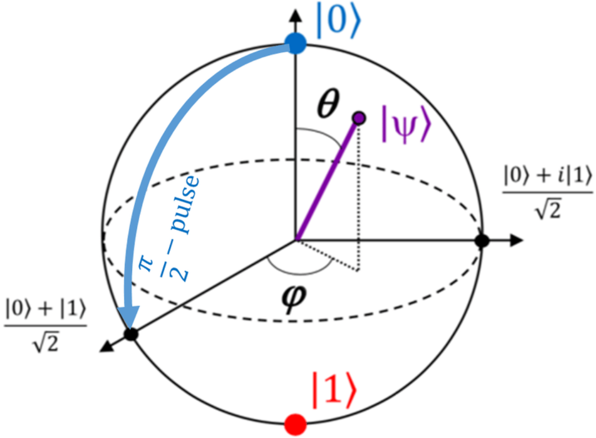
Quantum measurement

- So how do we work with qubit states?
- Measurement principle: we cannot directly get the statevector $|\psi\rangle$ of some quantum state. Measuring the state changes it!
- We have to *measure* it, collapsing it to one of two vectors in some basis (the computational basis, usually)
- To measure $|\psi\rangle$, take the dot products: $\langle\psi|0\rangle^2$ is the probability it collapses to 0, and $\langle\psi|1\rangle^2$ is the probability it collapses to 1
- Exercise: Find the probabilities of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ collapsing to 0 or 1

Representing statevectors

- By definition, a statevector has norm 1
- This leads to a geometrical intuition of what a statevector looks like
- In two dimensions, we can represent all vectors of norm 1 with a unit circle
- In quantum mechanics, we represent 3D statevectors using a unit sphere

The Bloch Sphere



$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$$

The Bloch Sphere

- Recall that an arbitrary statevector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a superposition of the two states $|0\rangle$ and $|1\rangle$ chosen as our ONB
- Thus, a qubit can be any point on the Bloch sphere if we properly choose α and β
- We do so by letting $\alpha = \cos\frac{\theta}{2}$ and $\beta = e^{i\phi}\sin\frac{\theta}{2}$, so that θ is the polar angle and ϕ is the azimuthal angle
- We now have a system of polar coordinates for describing statevectors of a qubit
- Just like with wavefunctions, the squared norms $|\alpha|^2$ and $|\beta|^2$ describe the probability of the qubit collapsing into either of our two basis states

Back to Reality

How does this relate to actual qubits?

- Physical qubits are things like electrons or photons, which have certain properties that are described via quantum states, e.g. the spin of an electron
- We need the mathematics of quantum mechanics to be able to describe these qubits, and how we can mess with them
- The statevector tracks the quantum state of a physical qubit
- Our allowed operations are matrices that don't change the length of a vector: the special unitary group SU(n)
  - Spoiler: quantum gates are exactly these matrices!

Section 3

Qubit errors and why it's so hard to correct them

What is an error?

- In classical computing, an error on a bit flips that bit
- In quantum computing, a qubit can be in an infinite number of positions!
- An error is *any* perturbation of a qubit's state
- $|0\rangle$ could become $|1\rangle$ or $|+\rangle$, or with a smaller perturbation, $\frac{1}{\sqrt{6}}\begin{pmatrix}5\\1\end{pmatrix}$

Errors on the Bloch Sphere

- Infinitely many positions on the Bloch Sphere mean infinitely many possible errors!
- In real quantum computers, small interactions with the environment can cause the qubit to "drift" on the Bloch Sphere
- A *partial bit flip error* rotates the statevector about the x-axis
- A *phase flip error* rotates the statevector about the z-axis
- We want to find a way to correct both types of errors

So how do we fix an error?

- An easy classical method is called a repetition code
- Add redundancy to a bit: 0 becomes 000
- If any one bit flips, we can spot it and fix it
- To protect against two bit flips, have four bits of redundancy: 0 becomes 00000
- Exercise: how many bits of redundancy do we need to protect against n bit flips?

Redundancy for qubits

- So we just need to add redundant qubits, right?
- Nope. There's an important theorem called the no-cloning theorem:

Theorem

If $|\psi\rangle$ is an arbitrary quantum state, we cannot make a copy of it. That is, we cannot go from $|\psi\rangle|0\rangle \to |\psi\rangle|\psi\rangle$.

- Note: $|\psi\rangle|0\rangle$ is the tensor product of $|\psi\rangle$ and $|0\rangle$; we represent multiple qubits in a system by taking their tensor product
- The no-cloning theorem rules out repetition codes entirely
- Sketch of proof: if we want to duplicate a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we need to know α and β. But we can't measure them!

Conclusion

- No-cloning theorem rules out basic error correction schemes
- We need something more sophisticated
- There are classical error correcting codes, that instead of using redundancy, use *parity bits*
- We'll use a similar idea, indirectly checking the qubits to make a bit-flip code. To be covered next week!

To imagine the quantum spin of a particle, imagine a spinning ball, except it's not a ball and it's not spinning.

— John Physics (1000 BC)

Bibliography

T.G. Wong.
Introduction to Classical and Quantum Computing.
Rooted Groove., 2022.