Inverse Ackermann Function

Ian Chen

Outline

The Function

RMQ Preprocessing

Disjoint Sets

Section 1

The Function

Motivation

This function is meant to capture different orders of growth.

1. linear
2. logarithmic
3. iterated logarithmic

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Tarjan's Definition

Definition (Tarjan '75)

$$A(i,j) = \begin{cases} 2j & \textbf{for } i = 0 \wedge j \geq 1 \\ 1 & \textbf{for } i \geq 1 \wedge j = 0 \\ A(i-1, A(i, j-1)) & \textbf{otherwise} \end{cases}$$

Definition (Tarjan '75)

$$\alpha(i,x) = \min \{\, j \mid A(i,j) \geq x \,\}$$

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Definition

$$\beta(i,j) = \begin{cases} 0 & \textbf{for } j = 1 \\ \lfloor \sqrt{j} \rfloor & \textbf{for } i = 1 \wedge j > 1 \\ 1 + \beta(i, \beta(i-1, j)) & \textbf{otherwise} \end{cases}$$

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A sequence of functions

- $\alpha(1, n) = \lfloor n/2 \rfloor$
- $\alpha(2, n) = \lg n$
- $\alpha(3, n) = \lg^* n$

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Section 2

RMQ Preprocessing

A Short but Cute Result

Suppose we are given an array of n integers. We want to be able to answer *range minimum queries* in few steps.

$$\lambda(2k, n) = \alpha(k, n) \qquad \lambda(2k+1, n) = \beta(k, n)$$

Theorem (Alon '87)

We can answer range minimum queries in k steps using $\mathcal{O}(nk\lambda(k, n))$ preprocessing space.

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Divide and Conquer

Let $T_k(n)$ be the time to preprocess an array of n elements for k-step queries.

$$T_2(n) = 2n \lg n$$

$$T_3(n) = 3n \lg \lg n$$

$$T_k(n) = \frac{n}{\lambda(k-2,n)} T_k(\lambda(k-2,n)) + T_{k-2}(n/\lambda(k-2,n)) + 2n$$

$$T_k(n) = \mathcal{O}(nk\lambda(k,n))$$

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Additional Results

- The analysis is tight.
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Section 3

Disjoint Sets

Union-Find

Given a set of elements, label them 1 through n. We want to partition these elements into buckets, supporting the operations

- Union
- Find

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Implementation

$\underline{\text{FIND}(x)}$:

$y \leftarrow x$

while $y \neq parent(y)$

  $y \leftarrow parent(y)$

$\text{COMPRESS}(x, y)$

return x

$\underline{\text{COMPRESS}(x, y)}$:

if $x \neq y$

  $\text{COMPRESS}(parent(x), y)$

  $parent(x) \leftarrow parent(y)$

Implementation

```
FIND(x):
    y ← x
    while y ≠ parent(y)
        y ← parent(y)
    COMPRESS(x, y)
    return x
```

```
COMPRESS(x, y):
    if x ≠ y
        COMPRESS(parent(x), y)
        parent(x) ← parent(y)
```

Implementation

```
UnionLeader(x, y):
    if rank(x) > rank(y)
        leader(y) ← x
    else
        leader(x) ← y
        if rank(x) = rank(y)
            rank(x) ← rank(x) + 1
```

```
Union(x, y):
    x̄ ← Find(x)
    ȳ ← Find(y)
    UnionLeader(x̄, ȳ)
```

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Analysis

1. leader ranks only increase
2. $parent(x)$ has lower rank than x
3. $size(\overline{x})$ is at least $2^{rank(\overline{x})}$
4. For any rank r, there are at most $n/2^r$ elements of rank r

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Analysis

1. Amortized analysis
2. Number of pointer operations
3. Find is dominated by Compress
4. can make all calls to UnionLeader before Compress
5. can Shatter the tree into two forests

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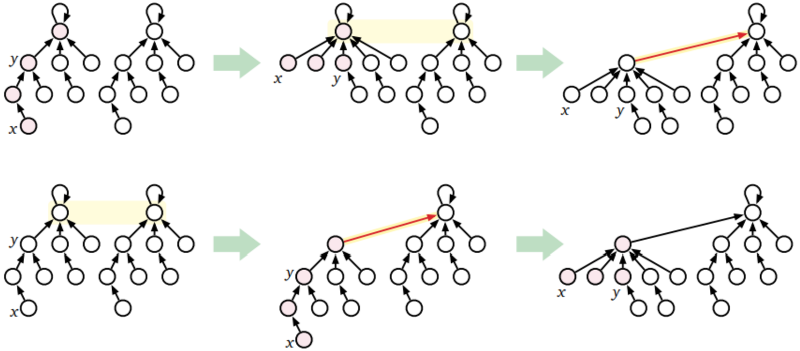
Compress and Shatter

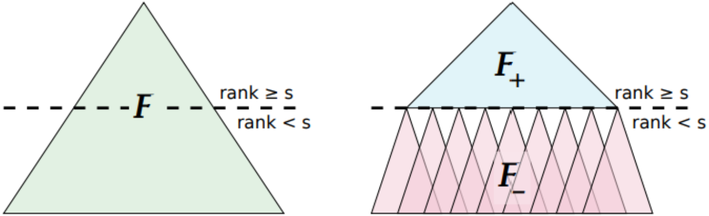
```
COMPRESS(x, y, F):
    if rank(x) ≥ s
        COMPRESS(x, y, F₊)
    else if rank(y) < s
        COMPRESS(x, y, F₋)
    else
        z ← x
        while rank(z) < s
            z′ ← parent(z)
            parent(z) ← z
            z ← z′
        parent(z) ← z
        COMPRESS(parent(z), y, F₊)
```

Analysis



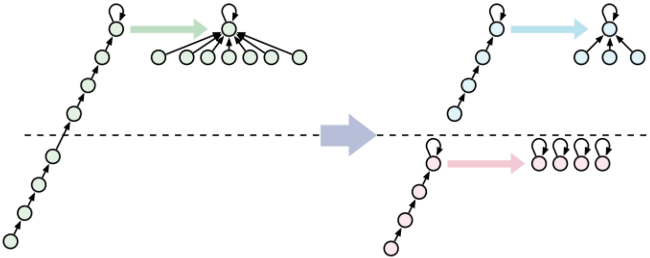
Top row: A COMPRESS followed by a UNION. Bottom row: The same operations in the opposite order.

Analysis



Splitting the forest F (in this case, a single tree) into sub-forests F_+ and F_- at rank s.

Analysis



Analysis

Let $T(m, n, r)$ be the number of pointer operations that m Compress operation takes, on a tree with n nodes and maximum rank r.

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$$T(m, n, r) \leq nr$$

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Consider the sequences of m_+ and m_- COMPRESS calls on F_+ and F_-.

$$T(m,n,r) \leq T(m_+,n_+,r) + T(m_-,n_-,r) + m_+ + n$$

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Analysis

Let $s = \lg r$.

Since $n_+ < n/2^s$, we have that

$$T(m_+, n_+, r) \leq r n_+ \leq rn/2^s = n$$

$$T(m, n, r) \leq T(m_-, n_-, \lg r) + m_+ + 2n$$

Letting $T'(m, n, r) = T(m, n, r) - m$,

$$T'(m, n, r) \leq T'(m, n, \lg r) + 2n$$

Theorem

$$T(m, n, r) \leq m + 2n \lg^* r$$

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Analysis

Let $s = \lg^* r$.

Since $n_+ < n/2^s$, we have that

$$T(m_+, n_+, r) \leq m_+ + 2n_+ \lg^* r \leq m_+ + 2n \frac{\lg^* r}{2^{\lg^* r}} \leq m + 2n$$

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$$T'(m, n, r) \leq T'(m, n, \lg^* r) + 3n$$

Theorem

$$T(m, n, r) \leq 2m + 3n \lg^{**} r$$

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Theorem

$$T(m,n,r) \leq cm + (c+1)n \lg^{*^c} r$$

$$T(m,n,r) \leq cm + (c+1)n\alpha(c,r)$$

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$$T(m,n,r) \leq m\alpha(n)$$

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Questions?

Brainteaser (Erickson)

Consider the following game. I choose a positive integer n and keep it secret; your goal is to discover this integer. We play the game in rounds. In each round, you write a list of at most n integers on the blackboard. If you write more than n numbers in a single round, you lose. If n is one of the numbers you wrote, you win the game; otherwise, I announce which of the numbers you wrote is smaller or larger than n, and we proceed to the next round.

Describe a strategy that wins in $\mathcal{O}(\alpha(n))$ rounds.

WAGA WAGA

— Sariel Har-Peled (2024)

All problems in computer science can be solved by another level of indirection.

— David Wheeler (2014)

Bibliography I

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