Nim

Rules of Nim

- There are 2 players, Player 1 and Player 2
- There are n piles of stones $a_1, a_2, \ldots, a_n$
- The players take turns making moves, with Player 1 starting
- On each turn, a player removes a positive number of stones from any non-empty pile
- Game end: when there are no stones left
- Winner: last player to make a move
- Let’s try playing Nim!

Questions we want to answer about Nim

- Given a starting game state, can one of the players guarantee a win?
  - Answer: Zermelo's theorem – In a finite two-person game with perfect information, at least one of the players can force a draw. If the game cannot end in a draw, exactly one of the players can force a win.
- Can we determine who wins?
- If so, how can the expected winner play optimally to guarantee their win?

Determining who wins

- For a game state g, let integer G be a "Nim-value" representing whether it is a winning or losing game state
- Let the Nim-value of a singular pile be the number of stones in that pile
- Given game states a, b, let $a + b$ be the game that includes the piles of a and b
- Let $A \bigoplus B$ be the Nim-value of $a + b$
- Hopefully the operator $\bigoplus$ exists, and let's try and find it

Properties of $\bigoplus$

- $(A \bigoplus B) \bigoplus C = A \bigoplus (B \bigoplus C)$
- $A \bigoplus B = B \bigoplus A$
- $A \bigoplus 0 = A$
- $A \bigoplus A = 0$
- Any guesses on what $\bigoplus$ could be?
- Xor!!!
- Claim: For a nim game with n piles $a_1, a_2, \dots, a_n$, the game state is winning if $a_1 \bigoplus a_2 \bigoplus \cdots \bigoplus a_n > 0$ and losing otherwise.

Proof of the Claim

- The empty game has Nim-value 0
- A non-empty one pile game has Nim-value > 0
- In a general game with n piles $a_1, a_2, \dots, a_n$,
  - If our current Nim-value is 0 and we take s stones from a_i, our new Nim-value is $a_1 \oplus a_2 \oplus \cdots \oplus a_n \oplus a_i \oplus (a_i - s) \neq 0$
    - $a_i \oplus (a_i - s) = 0$ iff $a_i = a_i - s$. Contradiction.
  - If our current Nim-value is non-zero, we can take s stones from a_i such that $a_i \oplus (a_i - s) = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ stones and the new Nim-value will be 0
    - Suppose the gth bit is the greatest significant bit of $a_1 \oplus a_2 \oplus \cdots \oplus a_n$. There exists a_i which has gth bit $= 1$. We can take s stones from a_i such that the above equation is satisfied

Sprague-Grundy Function

- $SG(X) = 0$ for terminal state X
- Otherwise, $SG(X) = mex\big(SG(X_1), SG(X_2), \ldots, SG(X_k)\big)$
  - $X_1, X_2, \ldots, X_k$ are game states that can be reached from X in one move
  - mex (minimum excluded) returns the smallest non-negative integer not in the set
- If $SG(X) = 0$, X is a losing game state ⇒ Player 2 wins
- If $SG(X) \neq 0$, X is a winning game state ⇒ Player 1 wins

Sprague-Grundy Theorem

- Let G be a game that consists of independent sub-games $G_1, G_2 \ldots, G_n$. That is, for each move in G, we choose a sub-game and make a move
  - Then, $SG(G) = SG(G_1) \oplus SG(G_2) \oplus \cdots \oplus SG(G_n)$
  - Just like in Nim!

	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	2	5
2	2	3	0	1	6
3	3	2	1	0	7
4	4	5	6	7	0

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  - For example, $SG(3,2) = SG(3) \oplus SG(2)$

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4	4	5	6	7	0

- For example, $SG(3,2) = mex(SG(0,2), SG(1,2), SG(2,2), SG(3,0), SG(3,1))$

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3	3	2	1	0	7
4	4	5	6	7	0

  - In fact, any impartial game that can be represented as a DAG can be translated into an equivalent Nim position

Variations of Nim

- Now that we know the strategy to solve impartial games, let’s play some games!

Addition Nim

- Same rules as Nim
  - With an extra move: at any move, you may choose to add any number of stones to any one pile
  - The “addition move” can be done a finite number of times

Solution to Addition Nim

- The game is the same as Nim: for any given game state, whoever wins in Nim also wins in Addition Nim
- The winning player uses the same strategy as Nim, only deviating when the losing player plays an “addition move”
  - The winning player can simply undo the “addition move” by removing the added stones
  - Key: “addition move” can be done a **finite** number of times

Square Nim

- Same rules as Nim
  - Except: the number of stones you remove must be a square number

Solution to Square Nim

- Don't think too hard – just calculate SG values!

```
SG[0] = 0;
for (int i = 0; i < n; i++) {
    vector<int> next_moves;
    for (int j = 1; j * j <= n; j++) {
        next_moves.push_back(SG[i - j * j]);
    }
    SG[i] = mex(next_moves);
}
```

- $O(N\sqrt{N})$

Restricted Nim

- Same rules as Nim
  - Except: Number of stones removed is at most k

Solution to Restricted Nim

- Consider one pile first and let $k = 3$

No. of stones	0	1	2	3	4	5	6	7	8
W/L	L	W	W	W	L	W	W	W	L
SG	0	1	2	3	0	1	2	3	0

- When we have n piles, the SG value is $(p_1\%(k+1))\oplus(p_2\%(k+1))\oplus\ldots\oplus(p_n\%(k+1))$

Arbitrary Nim (Atcoder)

- Same rules as Restricted Nim
  - Twist: we get to choose k such that player 1 wins, where k is the maximum number of stones we can remove
    - If k does not exist, print 0
    - If k exists, determine if there is a maximum k
      - If maximum exists, print maximum k
      - If maximum does not exist, print -1

Solution to Arbitrary Nim

- Consider $G = p_1 \oplus p_2 \oplus \ldots \oplus p_3$
- If $G \neq 0$, we can make k infinitely large and $(p_1\%(k+1)) \oplus (p_2\%(k+1)) \oplus \ldots \oplus (p_n\%(k+1)) = G \neq 0$
  - Print -1
- If $G = 0$, let $count(x)$ be the number of piles with exactly x stones
  - If all $count(x)$ are even, player 2 always wins regardless of k (mirroring)
  - Otherwise, let p be the largest pile such that $count(p)$ is odd
    - We can pick $k = p - 1$
    - $(p_1\%(k+1)) \oplus (p_2\%(k+1)) \oplus \ldots \oplus (p_n\%(k+1)) = G \oplus p \neq 0$

Questions?

Brainteaser

- Given an uncolored 6x6 grid, choose 5 squares to color. Then, repeatedly color any square that has at least 2 neighboring colored squares. Can you choose the 5 initial squares such that the entire grid eventually gets colored? If not, why?

References

- https://codeforces.com/blog/entry/66040
- https://atcoder.jp/contests/arc168/tasks/arc168_b