Dynamics and Chaos

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Outline

Section 1

A Motivating Example

Damped Driven Pendulum

Newton's Second Law for the damped driven pendulum:

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 \sin\theta = f\cos(\omega t),$$

where $\omega_0^2 = \frac{g}{l}$ and $f = \frac{F}{ml}$.

We can nondimensionalize (via the Buckingham Π theorem) the differential equations using the following:

$$\begin{cases} t := \omega_0 t \\ \frac{1}{q} := \gamma/\omega_0 \\ f := f/\omega_0^2 \\ \omega := \omega/\omega_0 \end{cases}, \text{ which gives } \ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin\theta = f\cos(\omega t).$$

A Small Angle Approximation

- For small θ, $\sin\theta \approx \theta$, so the differential equation becomes $\ddot{\theta} + \frac{1}{q}\dot{\theta} + \theta = f\cos(\omega t)$.
- We know that the solution to this differential equation is the sum of a transient solution that depends on initial conditions and a steady state solution that is independent of initial solutions.

Section 2

Formal Theory

Definitions

Definition

(X, d) metric space. $f : X \to X$.

1. A **fixed point** $x \in X$ is a point s.t. $f(x) = x$.
2. A **periodic point** $x \in X$ is a point s.t. $\exists n \in \mathbb{N} : f^n(x) = x$, where the least such n is called the **prime period**.
3. The **orbit** of x under f (denoted $O_f(x)$) is the sequence $x, f(x), \cdots, f^n(x), \cdots$ if f not invertible or $\cdots, f^{-1}(x), x, f(x), \cdots$ if f invertible.

Definitions (cont.)

Definition

$f : X \to X$ **exhibit sensitive dependence on initial conditions** if $\exists \delta > 0$ s.t.
$\forall x \in X, \epsilon > 0 \exists y \in X, N \in \mathbb{N} : x \neq y, d(x, y) < \epsilon, d(f^N(x), f^N(y)) \geq \delta.$

Rmk. If a system possesses sensitive dependence, then numerical computation becomes invalid, because the inaccuracies of numerical computation become amplified. The true orbit may diverge from the computed orbit.

Topological Transitivity

Definition

$f : X \to X$ is **topologically transitive** if $\exists x \in X$ s.t. $O_f(x)$ is dense in X.

Proposition

If X has no isolated points, $f : X \to X$ continuous, then f topologically transitive is equivalent to the following: for nonempty open sets $U, V \subset X$, then $\exists N \in \mathbb{N} : f^N(U) \cap V \neq \emptyset$.

Rmk. Reverse direction is obvious (consider contrapositive). The following is a proof of the forward direction.

Proof of Proposition (optional)

Proof

$\because f$ topologically transitive $\therefore \exists n, m \in \mathbb{Z} : f^n(x) \in U, f^m(x) \in V$. Note that $f^n(x) \in U \to x \in f^{-n}(U) \to f^m(x) \in f^{m-n}(U)$, hence $f^m(x) \in f^{m-n}(U) \cap V \neq \emptyset$. If $m > n$, we are done since $m - n \in \mathbb{N}$. Suppose $\nexists m > n : f^m(x) \in V$. Then there exists decreasing sequence $(n_k)_{k=1}^{\infty} \subset \mathbb{Z}$ s.t.

1. $n_k \leq n \forall k$
2. $n_k \to -\infty$ as $k \to \infty$
3. $f^{n_k}(x) \in V \forall k$ by denseness of $O_f(x)$ in X
4. $f^{n_k}(x) \to f^m(x)$ as $k \to \infty$

$\because f^m(x) \in f^{m-n}(U) \therefore$ by (4) $\exists m' \in (n_k) : m' < 2m - n, f^{m'}(x) \in f^{m-n}(U)$. Pick $x' := f^{n-m}(f^{m'}(x)) \in f^{n-m}(f^{m-n}(U)) = U$. Then $f^{2m-n-m'}(x') = f^m(x) \in V$. Take $N = 2m - n - m' \in \mathbb{N}$. Then $x' \in U \to f^N(x') \in f^N(U)$, and $f^N(x') \in V$, hence $f^N(U) \cap V \neq \emptyset$. ■

Chaos

Definition (Devaney)

$f : X \to X$ continuous. f is **chaotic** if it is topologically transitive and its periodic points are dense.

Theorem

Chaotic maps exhibit sensitive dependence on initial conditions, unless the entire space consists of a single periodic orbit.

Rmk. The converse to the theorem is not true. A common misunderstanding is that chaotic is equivalent to sensitive dependence on initial conditions. But this is obviously false from definition of chaotic.

Proof of Theorem (Optional)

Proof

We will prove sensitive dependence for $\epsilon \in (0, \delta)$. Then the case $\epsilon \geq \delta$ follows immediately.

Unless the entire space consists of only a single periodic orbit, denseness of periodic points implies that there exists at least 2 distinct periodic orbits. Because periodic orbits are disjoint, $\exists p, q$ on separate periodic orbits s.t. $\delta := \frac{1}{8} \min_{n,m \in \mathbb{Z}} (d(f^n(p), f^m(q)) > 0$. We claim δ works.

Obs. Fix $x \in X$. Then the orbit of either p or q is always at least a distance of 4δ from x. WLOG this point is q.

Take $V := \bigcap_{i=0}^{n} f^{-i}(B^o_\delta(f^i(q)))$. By previous proposition,

$\exists k \in \mathbb{N} : f^k(B^o_\epsilon(x)) \cap V \neq \emptyset \rightarrow \exists y \in B^o_\epsilon(x) : f^k(y) \in V$.

Proof of Theorem (cont.)

Proof

Suppose $p \in B^o_\epsilon(x)$ and has period n. Consider
$N := n(\lfloor \frac{k}{n} \rfloor + 1) \to 0 < N - k \leq n$.
By triangle inequality: $d(f^N(p), f^N(y)) = d(p, f^N(y)) \geq d(x, f^{N-k}(q)) - d(f^{N-k}(q), f^N(y)) - d(p, x) \geq 4\delta - \delta - \delta = 2\delta$, because $p \in B^o_\epsilon(x) \subset B^o_\delta(x)$ and $f^N(y) \in f^{N-k}(V) \subset B^o_\delta(f^{N-k}(q))$ (take $i = N - k$ in definition of V).
Thus, either $d(f^N(p), f^N(x)) \geq \delta$ or $d(f^N(y), f^N(x)) \geq \delta$.
Recall that $d(p, x) < \epsilon$ and $d(y, x) < \epsilon$. ■

Example of Chaotic Mapping

Consider $f : [-2,2] \to [-2,2], x \mapsto 2|x| - 2$.

- Denseness of periodic points: we can always find subinterval I of length $1/2^{n-2}$ s.t. $f^n(I) = [-2,2]$. Thus, f^n has a fixed point on I.
- Topological transitivity: $\forall U \subset [-2,2], \exists n \in \mathbb{N} : f^n(U) = [-2,2]$. Then $\forall V \subset [-2,2], f^n(U) \cap V \neq \emptyset$.

Section 3

Lyapunov Exponent

Definition

Definition (Discrete System)

$f : X \to X$. Let x, y be initial points in the phase space. The separation vector $\delta(n)$ is given by $\delta(n) = f^n(x) - f^n(y)$. The **Lyapunov exponent** λ is defined as

$$\lambda = \lim_{n \to \infty} \lim_{||\delta(0)|| \to 0} \frac{1}{n} \ln \frac{||\delta(n)||}{||\delta(0)||}.$$

Definition (Continuous System)

$f : \mathbb{R} \to X$. Let x, y be two trajectories in the phase space. The separation vector $\delta(t)$ is given by $\delta(t) = x(t) - y(t)$. The **Lyapunov exponent** λ is defined as

$$\lambda = \lim_{t \to \infty} \lim_{||\delta(0)|| \to 0} \frac{1}{t} \ln \frac{||\delta(t)||}{||\delta(0)||}.$$

Remarks

- Intuitively, this means that (for a discrete system) for n large, $||f^n(x+\delta) - f^n(x)|| \approx \delta e^{\lambda n}$.
- The Lyapunov Exponent measures the rate at which two initially extremely close states grow over time.
- If $\lambda > 0$, the distance between the two states grows over time, and is a good indicator of chaotic systems.
- From this, it is obvious that $\lambda > 0$ implies that f has sensitive dependence on initially conditions.
- However, $\lambda > 0$ does not *necessarily* imply chaos. Additional information regarding the system is needed.

Maximal Lyapunov Exponent

- λ may be different for different initial separation vectors $\delta(0)$, creating a spectrum.
- For example, consider the first order linear system $\vec{x}' = A\vec{x} + \vec{f}(t)$, with A constant. From differential equations, we can observe that real parts of eigenvalues of A are Lyapunov exponents.
- We define the **maximal Lyapunov exponent** as the largest value in the spectrum.

A Practical Formulation

- From $|f^n(x+\delta) - f^n(x)| \approx \delta e^{\lambda n}$, isolate λ: $\lambda \approx \frac{1}{n} \ln \left| \frac{f^n(x+\delta)-f^n(x)}{\delta} \right|$.
- Take $\delta \to 0$, we apply definition of a derivative: $\lambda \approx \frac{1}{n} \ln |(f^n(x))'|$.
- Using chain rule:

  $$(f^n(x))' = f'(f^{n-1}(x)) \cdot f'(f^{n-2}(x)) \cdots f'(f(x)) \cdot f'(x) = \prod_{i=0}^{n-1} f'(x_i),$$

  where $x_i := f^i(x)$. Substitute in $(f^n(x))'$: $\lambda \approx \dfrac{1}{n} \ln \left| \displaystyle\prod_{i=0}^{n-1} f'(x_i) \right|$.
- Take $n \to \infty$: $\lambda = \displaystyle\lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n} f'(x_i) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln |f'(x_i)|$.
- For continuous systems, an equivalent formulation is $\lambda = \int f(x) \ln |f'(x)| dx$.

Example

- Consider the doubling map on the unit circle: $D : S^1 \to S^1$ (equivalently $D : [0, 2\pi) \to [0, 2\pi)$), $\theta \mapsto 2\theta \mod 2\pi$, where S^n denotes the unit sphere in $(n+1)$-dimensional Euclidean space.
- $D'(\theta) = 2 \forall \theta \neq \pi$. We can ignore the discontinuity at $\theta = \pi$.
- Hence, $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln |f'(x_i)| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln 2 = \ln 2 > 0$, meaning D has sensitive dependence on initial conditions.
- Exercise: prove that D is chaotic.

Section 4

Bifurcations

Attraction and Repulsion

Definition

A fixed point x_0 is **attracting/stable** if $\exists \lambda \in (0,1)$ and open interval U containing x_0 s.t. $|f(x) - f(x_0)| = |f(x) - x_0| \leq \lambda |x - x_0| \forall x \in U$. Similarly, x_0 is **repelling/unstable** if $\exists \lambda > 1$ and open interval U containing x_0 s.t. $|f(x) - x_0| \geq \lambda |x - x_0| \forall x \in U$.

Rmk. It follows from the definition of attraction that $O_f(x) \to x_0 \forall x \in U$.

Definition

A periodic point of period n is attracting (repelling) if it is an attracting (repelling) fixed point of f^n.

A Practical Formulation

Theorem

$f : X \to X$ differentiable at fixed point $x_0 \in X$. Let $a := |f'(x_0)|$. Then:

1. x_0 is attracting iff $a < 1$.
2. x_0 is repelling iff $a > 1$.

Proof

We will prove (1). The proof of (2) is similar.
$(\leftarrow)$ $\because a = f'(x_0) \therefore \forall \epsilon > 0 \exists \delta > 0 : 0 < |x - x_0| < \delta \to |\frac{f(x)-f(x_0)}{x-x_0}| \in (a - \epsilon, a + \epsilon)$. Take $\epsilon < 1 - a$ and $U = (x_0 - \delta, x_0 + \delta)$.
$(\to)$ Consider contrapositive: $a \geq 1 \to$ not attracting.

Bifurcation

- Let λ be a external parameter. Let $f : X \to X$ be a function dependent on λ. Varying λ generates a family of functions, denoted f_λ, where each function in the family uses a different value of λ.
- As we change f by varying λ, there are certain points in the family where the qualitative behavior of the function changes. These changes are called **bifurcations**, and the values of the parameter λ where these changes occur are called **bifurcation points**.

Theorem: No Bifurcation

Theorem

Let f_λ be a family of functions. Suppose $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) \neq 1$. Then $\exists$ interval I s.t. $x_0 \in I$, interval N s.t. $\lambda_0 \in N$, and continuously differentiable function $p : N \to I$ s.t. $p(\lambda_0) = x_0$ and $f_\lambda(p(\lambda)) = p(\lambda)$. Moreover, f_λ has no other fixed points in I.

Proof

Consider function $g(x, \lambda) = f_\lambda(x) - x$. We have $g(x_0, \lambda_0) = 0$, $\frac{\partial g}{\partial x}(x_0, \lambda_0) = f'_{\lambda_0}(x_0) - 1 \neq 0$. Apply the implicit function theorem on g at (x_0, λ_0).

Period Doubling Bifurcation: Definition

Definition

A family of functions f_λ undergoes a **period doubling bifurcation** at $\lambda = \lambda_0$ if $\exists$ open interval I, $\epsilon > 0$ s.t.:

1. $\forall \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon], \exists!$ fixed point $p_\lambda \in I$ for f_λ.
2. $\forall \lambda \in (\lambda_0 - \epsilon, \lambda_0], f_\lambda$ has no cycles of period 2 in I and p_λ is attracting (resp. repelling).
3. $\forall \lambda \in (\lambda_0, \lambda_0 + \epsilon), \exists!$ 2-cycle q_λ^1, q_λ^2 in I that is attracting (resp. repelling). Meantime, fixed point p_λ is repelling (resp. attracting).
4. As $\lambda \to \lambda_0$, $q_\lambda^i \to p_{\lambda_0}$.

Section 5

A Second Look at Damped Driven Pendulum

Spontaneous Symmetry Breaking Bifurcation

$$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin\theta = f\cos(\omega t).$$

- Note that the pendulum equation is invariant under the transformation $\theta \mapsto -\theta, t \mapsto t - \pi/\omega$. Thus, if $\theta_1(t)$ is a solution, then $\theta_2(t) = -\theta_1(t - \pi/\omega)$ is also a solution.
- Physically, this means that the equations make no distinction between the left and right sides of the vertical $\theta = 0$.
- We say the solution is *symmetric* if $\theta_1(t) = \theta_2(t)$; otherwise, *spontaneous symmetry breaking* has occurred. This symmetry breaking produces an additional loop on the phase diagram.

Spontaneous Symmetry Breaking Bifurcation (cont.)

- At $q \approx 1.246$, the symmetry is broken.
- For $q < 1.246$, the asymptotic solution (limit cycle) is symmetric.
- For $q > 1.246$, there exists a pair of stable, asymmetric asymptotic solutions. The original limit cycle still exists but becomes unstable. Depending on initial conditions, the trajectory converges to one of the stable limit cycles.
- Demo!

Period Doubling Bifurcation

Demo!

Bifurcation Diagram

1. Find limit cycle for some q and plot it in the 2D phase plane.
2. Construct is a 1D subspace (called the *Poincare section*) of the 2D phase plane.
3. Take the intersection of the limit cycle and the Poincare section.
4. Plot data points where q is the x coordinate and θ is y coordinate.

Very Scuffed Bifurcation Diagram that I drew

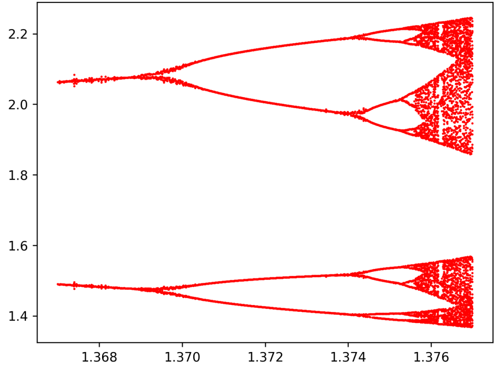

Figure: **Bifurcation Diagram of the Damped Driven Pendulum** with $f = 1, 5$, $\omega = 2/3$, and $1.367 < q < 1.377$.

Section 6

A Classic Example: Logistic Map

Definition

A 1D map can be expressed in the form of $x_{n+1} = f(x_n)$.

Definition

The **logistic map** is a family of functions
$f\mu : \mathbb{R} \to \mathbb{R}, x \mapsto \mu x(1 - x), \mu \in \mathbb{R}$.

Specifically, we will investigate $\mu > 0$.

First Period Doubling Bifurcation

- Recall that fixed point x_* is stable if $|f'(x_*)| < 1$ and unstable if $|f'(x_*)| > 1$. Bifurcations usually occur at the *marginally stable* point of $|f'(x_*)| = 1$.
- The initial fixed points for small μ are: $x = \mu x(1-x) \to x_* = 0, 1 - 1/\mu$. $f'_\mu = \mu(1-2x)$, so $x_* = 0$ is stable for $0 < \mu < 1$, and $x_* = 1 - 1/\mu$ is stable for $1 < \mu < 3$.
- At $\mu = 3$, $x_* = 1 - 1/3 = 2/3$, and $f'_\mu(x_*) = \mu(1 - 2x_*) = 3(1 - 2 \cdot 2/3) = -1$, which is a marginally stable point, causing the first bifurcation.
- The 2 periodic points (denoted x_0 and x_1) after the bifurcation must both satisfy $x = f_\mu(f_\mu(x)) = f^2_\mu(x)$, which is a quartic equation. Note that 2 roots are the initial fixed points $x_* = 0, 1 - 1/\mu$, so factor them out to get $x_1, x_2 = \frac{1}{2\mu}(\mu + 1 \pm \sqrt{(\mu+1)(\mu-3)})$.

Subsequent Period Doubling Bifurcations

- To find the value of μ at which a bifurcation occurs (going from period-2^{n-1} to period-2^n, we use $(f_\mu^{2^{n-1}})'(x_i) = -1$ for all fixed points x_i of $f_\mu^{2^{n-1}}$.
- Using the chain rule,
$$(f_\mu^{2^{n-1}})'(x_0) = f_\mu'(f_\mu^{2^{n-1}-1}(x_0))f_\mu'(f_\mu^{2^{n-1}-2}(x_0))\cdots f_\mu'(f_\mu(x_0))f_\mu'(x_0)$$
$$= f_\mu'(x_{2^{n-1}-1})f_\mu'(x_{2^{n-2}-2})\cdots f_\mu'(x_1)f_\mu'(x_0) = \prod_{i=0}^{2^{n-1}-1} f_\mu'(x_i) =$$
$$\prod_{i=0}^{2^{n-1}-1} \mu(1-2x_i) = -1.$$
- We know where the fixed points x_i are for period-2^{n-1}, so we can solve the equation as a function in μ.

Feigenbaum Constant

- It turns out that if we take the ratio of the space between consecutive period doubling bifurcation points of the logistic map, we approach a value known as the Feigenbaum constant $\delta = 4.669...$ (A006890 in OEIS).
- Let q_n be the value of q of the n-th bifurcation. Let $\delta_n := \frac{q_{n+1} - q_n}{q_{n+2} - q_{n+1}}$. Then $\delta := \lim_{n \to \infty} \delta_n$.
- This constant is *universal* for all 1D maps with a single locally quadratic maximum.

Questions?

Further Reading

- Theorems for sufficient conditions of saddle node bifurcations and period doubling bifurcations
- Topological conjugacy and structural stability
- Sharkovsky's Theorem
- Fractal dimension of strange attractors

Chaos is lawless behavior governed entirely by law.

— Ian Stewart (1989)

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