(The Lemma that is Not Burnside's)

An Introduction to Burnside's Lemma

@nebu

Outline

Section 1

Motivation

Consider the following problem

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$$2^4 = 16$$

Since we have 4 objects to color, with two choices for each one.

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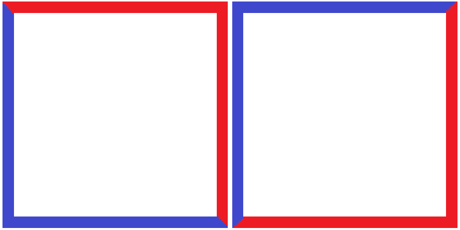
Are the following squares the "same" square or are they different?



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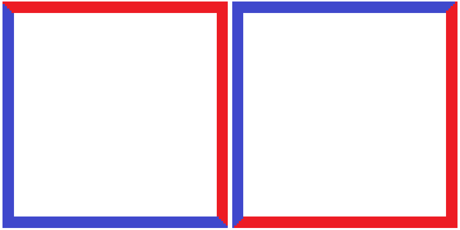


Under rotation by $\pi/2$, these are the same square.

Consider, then, this problem

The sides of a square are to be colored by either red or blue. How many different arrangements exist, if we treat colorings that can be obtained by rotation from another as identical?

Case-by-case solution

Permutation	Rotations different	Rotations identical
All sides red	1	1
All sides blue	1	1
One side red	4	1
One side blue	4	1
Two adjacent sides blue	4	1
Two opposite sides blue	2	1
Total	16	6

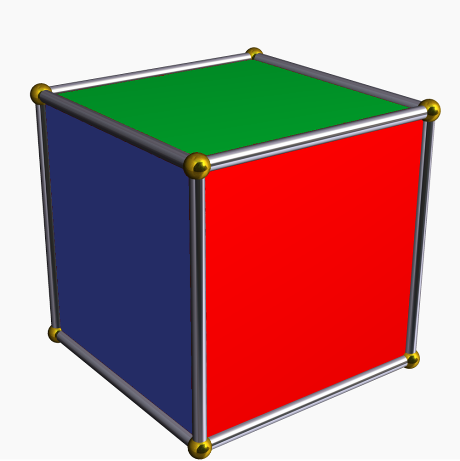
Extending the problem

What if there are now 3 colors?

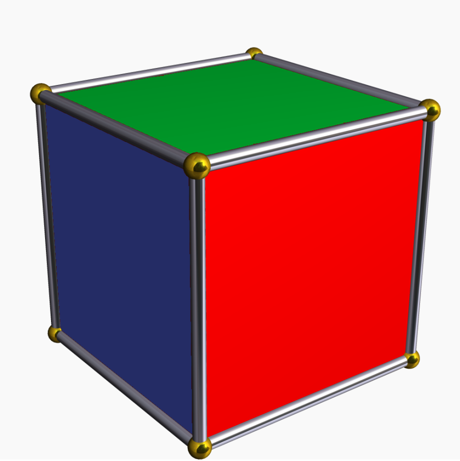
What if the shape is now a hexagon?

...etc.

What if we want to color a cube w/3 colors?



We need a generalization: Burnside's lemma!



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Intuitively, it provides a way to count distinct objects *up to* some equivalence relation, i.e., taking into account some symmetry.

Section 2

Groups

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- Existence of identity: There is a unique $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
- Existence of inverse: For all $a \in G$, there exists $a' \in G$ such that $a \cdot a' = a' \cdot a = e$.

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- Similarly, $(\mathbb{Q}, +)$ is a group.
- $(\mathbb{Z}_n, +)$, the group of integers modulo n under addition, is an abelian group.

Exercise

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- Is $(\mathbb{Z}, \cdot)$ a group?
- Is $(\mathbb{Q}, \cdot)$ a group?
- Is $(\mathbb{Q} \setminus \{0\}, \cdot)$ a group?
- (Trickier) Is $(\mathbb{Z}_n, \cdot)$ a group? What are its elements?

Section 3

Symmetry Groups

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Some intuition first.

Symmetry Groups

Consider two squares. They are part of the same equivalence class (i.e., we consider them the same square) if we can get one from the other by using:

- Rotation
- Reflection
- Translation
- Or a combination of the three.

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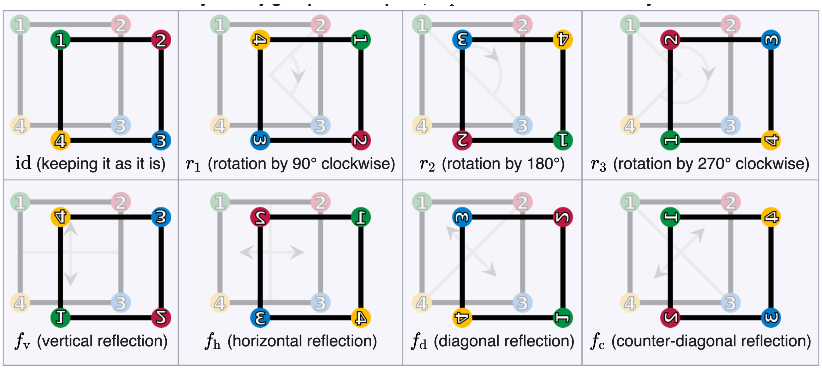
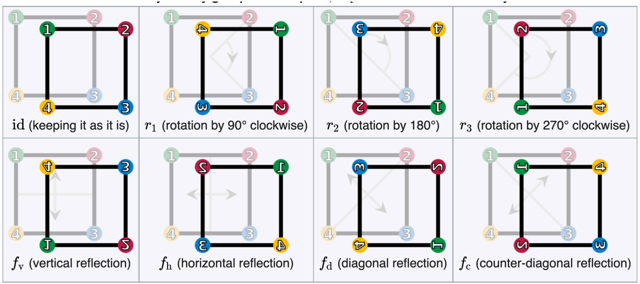
Visually



Set of symmetries: $(\mathrm{id}, r_1, r_2, r_3, f_v, f_h, f_d, f_c)$.

Visually



The set of symmetries are a set of functions. The functions are permutations of the vertices (1, 2, 3, 4).

Symmetries are Permutations are Bijective Functions

id is:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{array}$$

f_d is:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 4 \end{array}$$

What is the group operation?

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The group operation is thus function composition.
For instance, $f_h \circ r_3$ means:

- Rotate by $3\pi/2$.
- Reflect across the horizontal.

Turns out this is equivalent to f_d.

Is this really a group?

Check for yourself! (Spoiler: it is.)

Is it abelian?

Examples

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- The group we just looked at was D_4.
- There are many, many more...

Section 4

Actions of Symmetry Groups

Group Actions

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Satisfying

1. Identity: $\alpha(e, x) = x$ for all $x \in X$
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We often write gx instead of $\alpha(g, x)$, to get:

1. Identity: $ex = x$
2. Compatibility: $g(hx) = (gh)x$

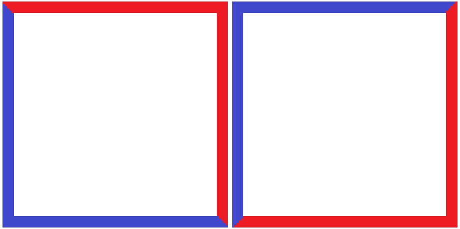
What are G and X

In the case of our square,

- G is the group of symmetries (C_4)
- X is the set of all possible colorings of the square.

Going back to our square...

This is the result of applying r_1 on the square:



Fixed Points

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The set of all fixed points of g is denoted $\mathrm{fix}(g)$ or (I dislike this notation) X^g:

$$\mathrm{fix}(g) = \{x \in X : gx = x\}$$

Orbits

For $x \in X$, an orbit is the set of elements to which we can move x via action by G:

$$Gx = \mathrm{orb}(x) = \{gx : g \in G\}$$

Orbits Partition X

Gx is clearly a subset of X. Consider $x' \in Gx$.

- It must be true that $Gx = Gx'$.
- By contradiction,
  - Let there be an element $y \in Gx'$ and $y \notin Gx$.
  - $y = g_1 x'$, but $x' = g_2 x \implies y = g_1(g_2)x \implies y = (g_1 g_2)x \implies y \in Gx$, for some $g_1, g_2 \in G$.

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- Thus, the concept of "number of orbits" of X makes sense.

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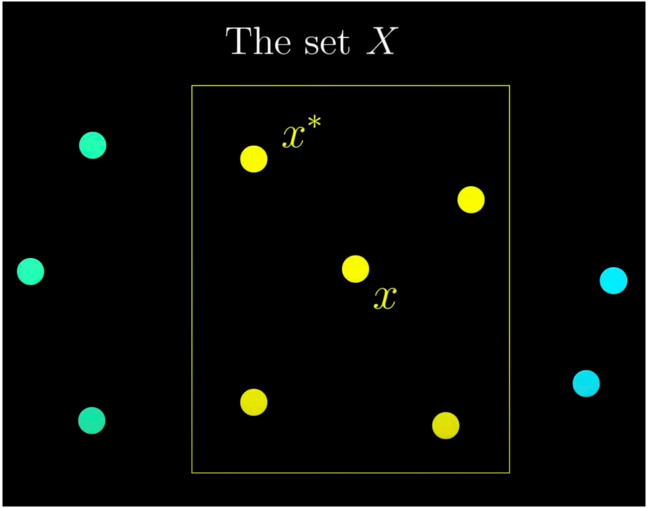
- The total number of orbits is the total number of colorings with the symmetry constraint.

Visually: Yellow Box is an Orbit (Identical Coloring)



One Last Thing: Stabilizers

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Contrast with fixed points of $g \in G$:

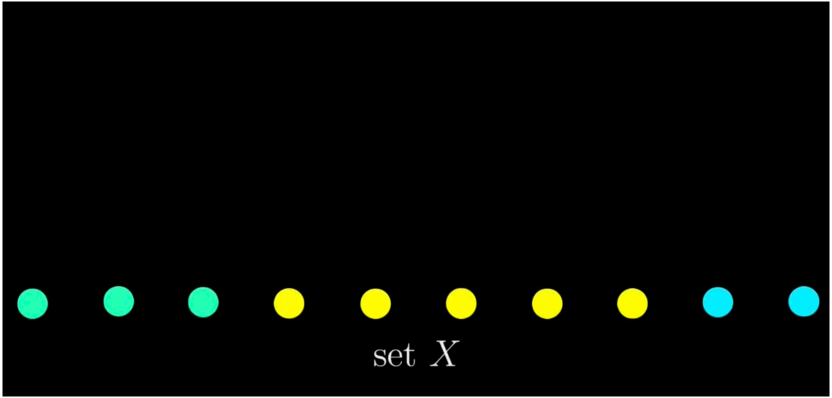
$$\text{fix}(g) = \{x \in X : gx = x\}$$

Section 5

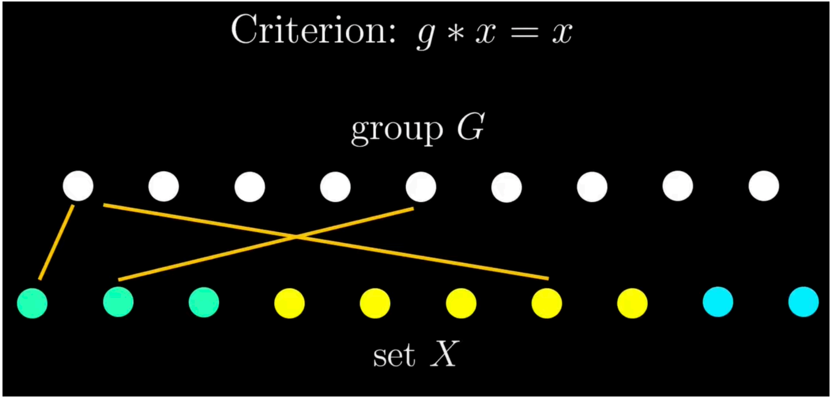
Counting Orbits and Burnside's Lemma

Line up the elements of X

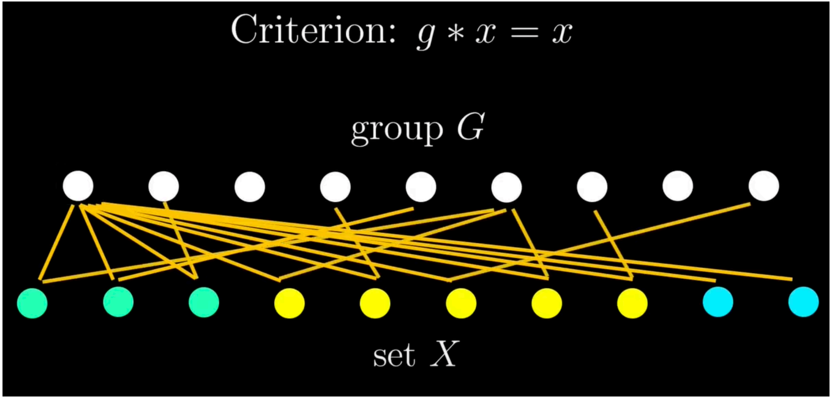
Same color means same orbit.



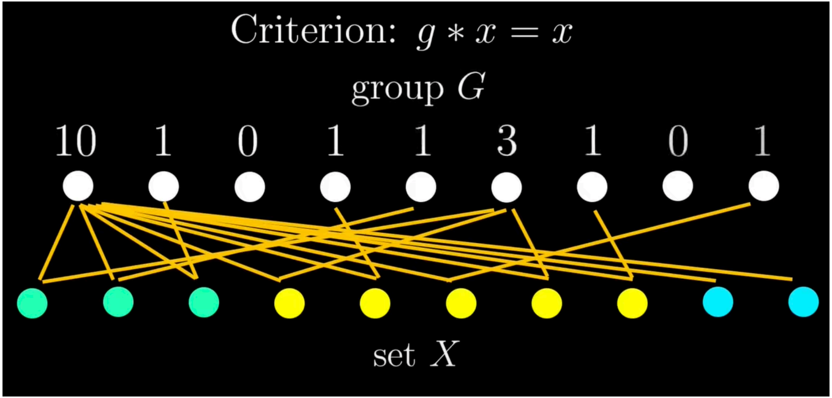
Draw G. Draw a line between g and x if $gx = x$.



Let's try to count the number of lines



We can count number of lines exiting each g



Formally

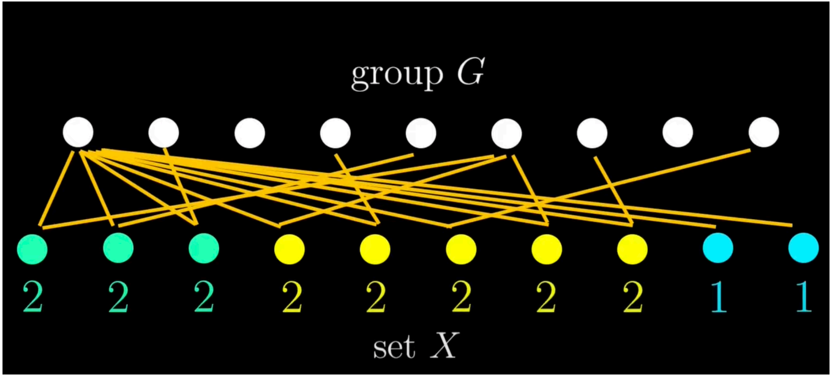
Recall,

$$\mathrm{fix}(g) = \{x \in X : gx = x\}$$

So our total number of spaghetti is

$$\sum_{g \in G} |\mathrm{fix}(g)|$$

But we can also count outgoing lines from each x



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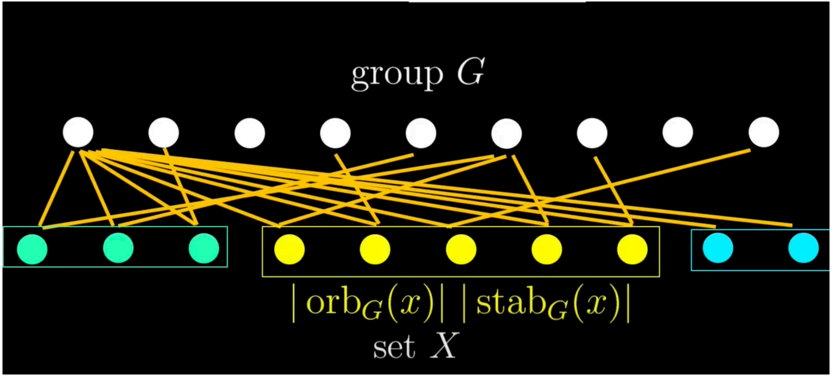
All x in the same orbit have the same number of stablizers, so the total number of outgoing spaghetti from an orbit is:

$$|\mathrm{stab}(x)||\mathrm{orb}(x)|$$

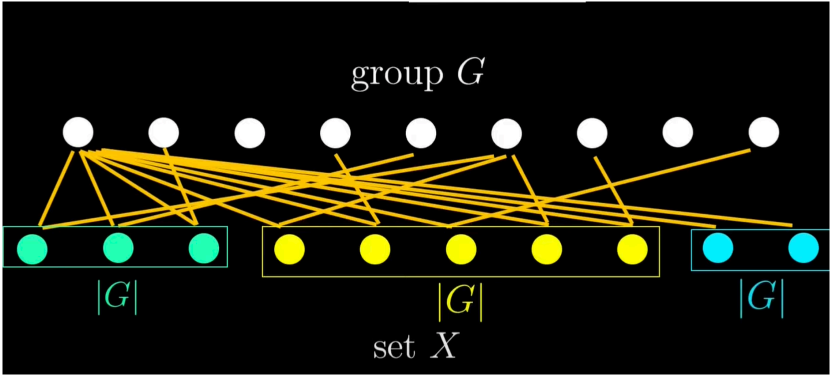
We then have



Orbit-Stabilizer Theorem

$$|G| = |\mathrm{stab}(x)||\mathrm{orb}(x)|$$

Thus



Putting it Together

$$\#\text{ of orbits}(|G|) = \sum_{g \in G} |\text{fix}(g)|$$

$$\implies \#\text{ of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

Burnside's Lemma

$$\boxed{\#\text{ of orbits} = \frac{1}{|G|}\sum_{g\in G}|\text{fix}(g)|}$$

Section 6

Examples of Usage

Original Problem

The sides of a square are to be colored by either red or blue. How many different arrangements exist, if we treat colorings that can be obtained by rotation from another as identical?

Using Burnside's Lemma

The group is C_4. Thus, $|G| = 4$.

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Applying Burnside's

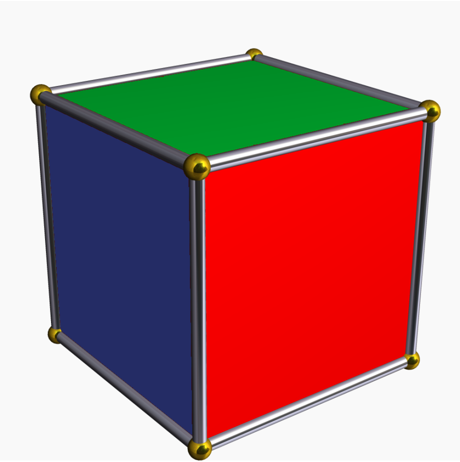
$$\#\text{ of orbits} = \frac{1}{4}\sum_{g\in G}|\text{fix}(g)|$$

$$\#\text{ of orbits} = \frac{1}{4}(16+2+2+4) = \frac{24}{4} = 6$$

Matches our casework!

Burnside's on the Cube

Want to color this cube with 3 colors. Rotations are the same cube.



Burnside's on the Cube

$|G| = 24$. The fixed points are:

- Identity: 3^6, all are fixed points.
- $\pi/2$ rotations: 4 lateral faces same color, can select axis faces. $2 \times 3 \times 3^3$ (accounting for $3\pi/2$ as well), along each axis.
- π: 2 uniquely colored lateral faces, top and bottom: 3×3^4 for each axis.
- $\pi/3$ Rotations about 8 diagonal axes: 8×3^2: each corner fixes a color.
- π Rotations about 6 edge midpoint axes: 6×3^3: each edge fixes a color for a pair of faces.

Burnside's on the Cube

Plugging into Burnside's, we get:

$$\# \text{ of orbits} = \frac{1}{24}\left(3^6 + 6\cdot 3^3 + 3\cdot 3^4 + 8\cdot 3^2 + 6\cdot 3^3\right) = 57$$

Credits

- Mathemaniac on YouTube for the graphics: `https://www.youtube.com/watch?v=6kfbotHL0fs`. The channel also has an excellent proof of the orbit-stabilizer theorem.