Ramsey's Theorem

Parth Deshmukh

Outline

Section 1

Friends and strangers

Six people walk into a party...

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- How do we represent this?
- Coloring the edges of graphs!

Into graph land

We represent each person as a vertex, and connect two people with a red edge if they know each other, and a blue edge if they don't.

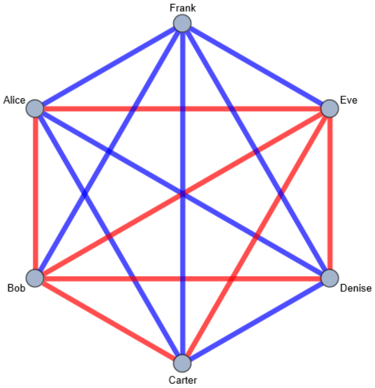

Figure: Our party - poor Frank doesn't know anyone :(

Friends and strangers

Theorem

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Friends and strangers

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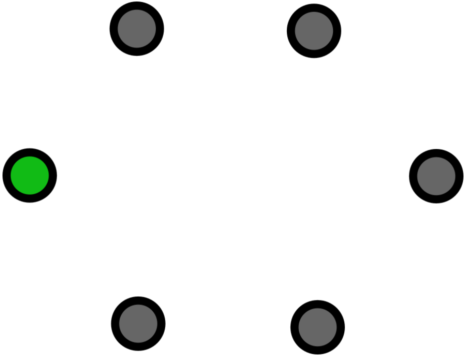
Formally:

Theorem

If we color the edges of a complete graph K_6 with two colors, there will be at least one monochromatic triangle as a subgraph.

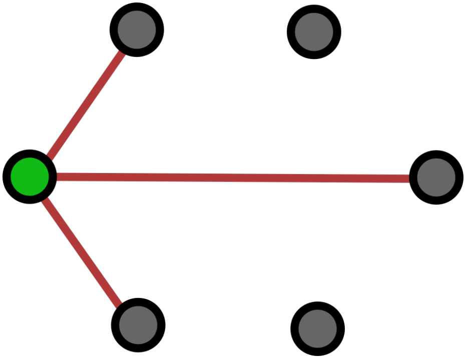
Proof of the theorem of friends and strangers

We will prove by contradiction. Our goal is to color a K_6 so it has no monochromatic triangle. Let's focus on one vertex, highlighted in green:



Proof of the theorem of friends and strangers

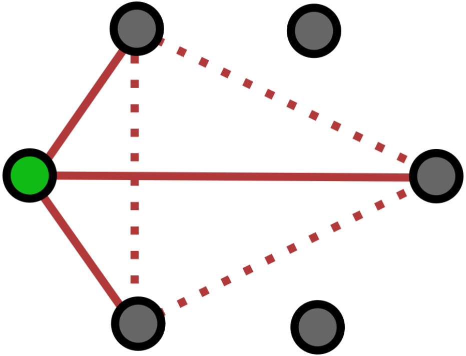
By the pigeonhole principle, at least 3 of this vertex's edges must be red:



If no such vertex exists, then every vertex has at least 3 blue edges; swap blue and red.

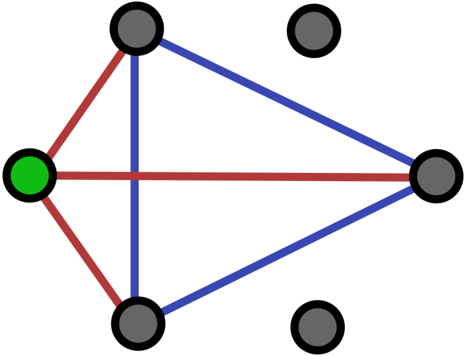
Proof of the theorem of friends and strangers

Now we need to connect those three neighbors. We can't connect them with a red edge, because that'd make a red triangle!



Proof of the theorem of friends and strangers

So we have to connect them in blue, making a blue triangle. Thus we can't avoid making a monochromatic triangle, proving the theorem.



Different numbers of people

- What if we have more than six people?

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- What if we have more than six people?
- If we have six or more people, this property holds because there's a copy of K_6 inside our representative graph $K_n, n >= 6$.

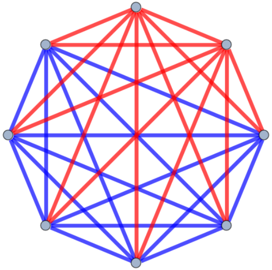

Figure: The subgraph marked by the blue edges is a copy of K_6.

Different numbers of people

- What if we have five people?

Different numbers of people

- What if we have five people?
- Looking at K_5, this property no longer holds:

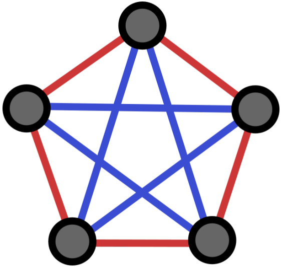

Figure: Each set of colored edges is a cycle of 5 elements, so no triangles!

Different numbers of people

Thus, the statement should really be “six *or more* people.” So this theorem really gives us a *threshold* of some sort. What is this threshold?

Section 2

Ramsey's Theorem

The full theorem

Theorem

Let integers $k, l > 2$*. Then there exists a minimum positive integer* $R(k, l)$ *so that if we color the edges of a complete graph with* $R(k, l)$ *vertices red or blue, there is either a monochromatic red clique of* k *vertices or blue clique of* l *vertices. [Bón06]*

- A "clique" is a complete subgraph.
- Our theorem of friends and strangers is really proving $R(3, 3) = 6$.

Proof sketch

- The proof of Ramsey's theorem uses induction.
- Note that $R(k, l) = R(l, k)$. We also have $R(k, 2) = k$ for the base cases.
- The inductive step is:

$$R(k, l) \leq R(k, l - 1) + R(k - 1, l)$$

Section 3

Calculating the Ramsey numbers

Just use a computer, right?

- Unfortunately, the time complexity grows far too fast.
- Suppose we want to check if $R(5, 5) = N$.
- A naive program would look like:

```
RAMSEY(5, 5, N):
    for every coloring C of K_N:
        if C does not contain a monochromatic 5-clique:
            output NO and the coloring C
    output YES
```

There might be better algorithms [Var], but there's not a lot of work on them; nothing seems to be fixing the time complexity just yet.

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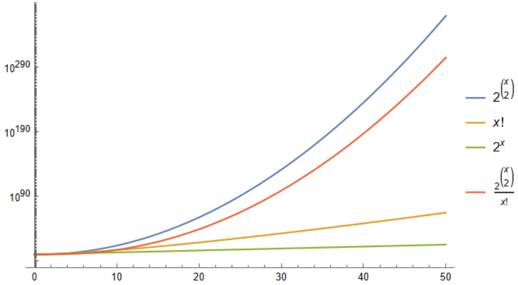

Figure: $43 \leq R(5,5) \leq 48$

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- Lower bound: [Bón06, 15.5]

$$(\sqrt{2})^k \leq R(k,k) \leq 4^{k-1}$$

- The proof for the lower bound uses the *probabilistic method*.

Section 4

The wide world of Ramsey theory

Ramsey-type theorems

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- Graham-Leeb-Rothschild theorem: Hales-Jewett theorem for subcubes instead of lines

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  - Van der Waerden and Schur are concerned with coloring the first N integers
- Those bounds tend to blow up *really* quickly and are even harder to compute
  - The proof that $\mathrm{S}(5) = 161$ took 2 petabytes of space!
  - `https://www.cs.utexas.edu/~marijn/Schur/`

Questions?

Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

— PAUL ERDOS ([ea90])

Σ

Bibliography

Miklós. Bóna.
A walk through combinatorics : an introduction to enumeration and graph theory.
World Scientific Pub., Hackensack, NJ, 2nd ed. edition, 2006.

Ronald L. Graham et al.
Ramsey Theory (2nd Ed.).
Wiley-Interscience, USA, 1990.

Various.
Algorithms for calculating r(5,5) and r(6,6).
https://mathoverflow.net/questions/210653/algorithms-for-calculating-r5-5-and-r6-6.
Accessed: 04-10-2023.