The PACE 2023 Challenge: Twin-Width

Anakin

Outline

Section 1

What is Twin-width?

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  - How easy is it to construct the object?
  - Algorithmic Complexity
  - Efficient encodings
  - *Decomposition*

Operations on Graphs: Disjoint Union

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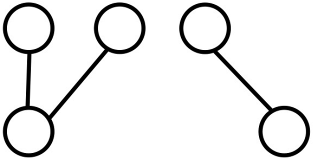
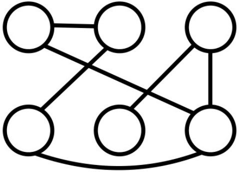


Operations on Graphs: Complement



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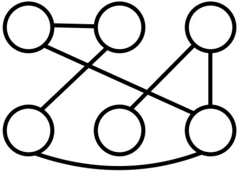

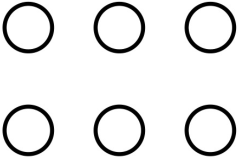

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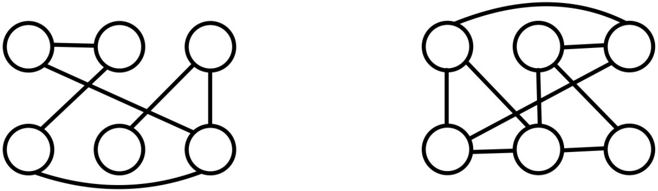

Cographs

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- K_1 is a cograph
- The disjoint union of two cographs is also a cograph
- The complement of a cograph is a cograph

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- This is called *twin-width* [BKTW20]

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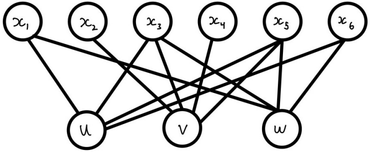
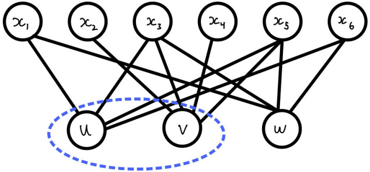
  - Our edges in E will have a color associated with them: black or red.
    - The red edges will be our *error* that we want
  - Vertices are black neighbors if linked by a black edge, and red neighbors if linked by a red edge

Contractions by picture





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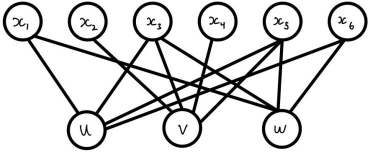

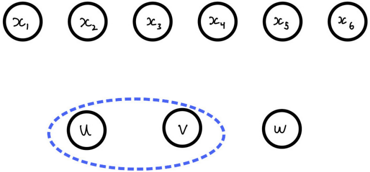

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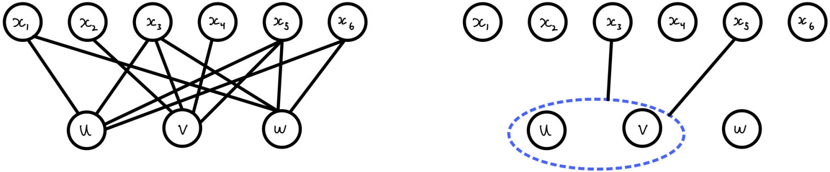

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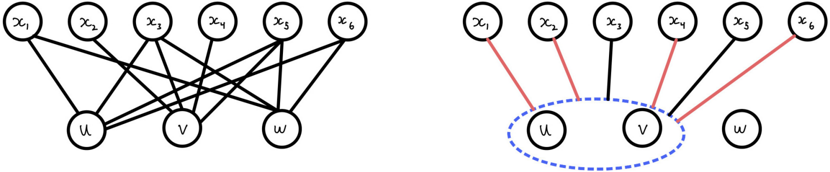

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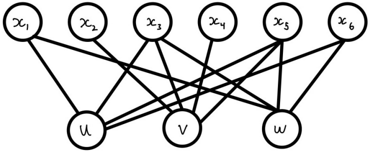

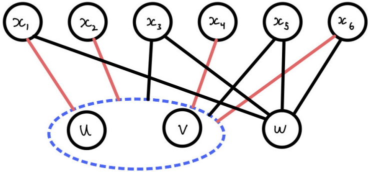

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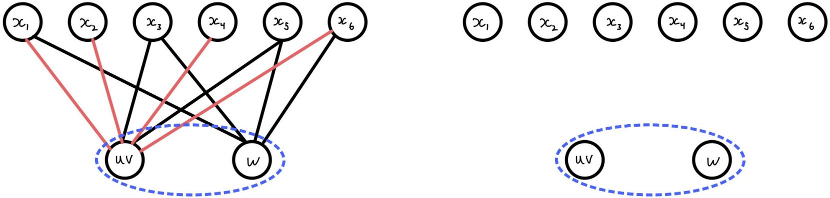

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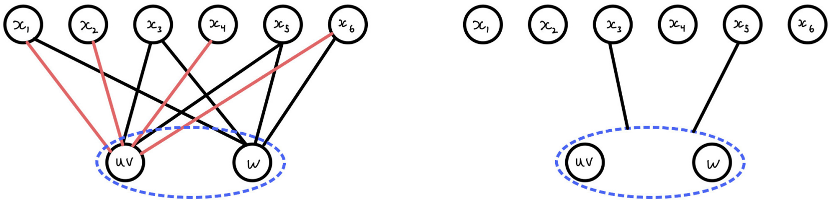

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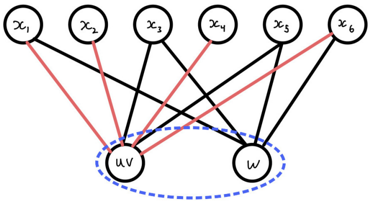

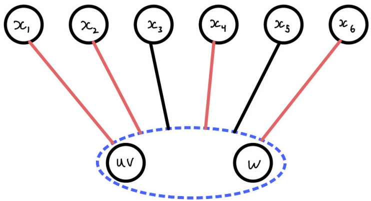

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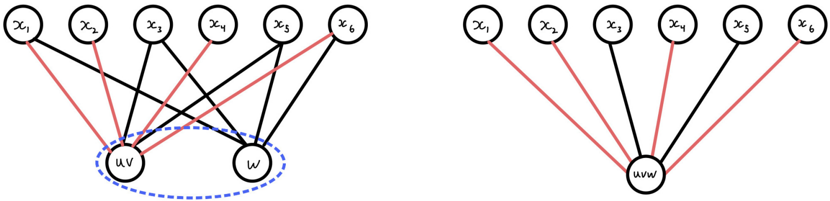

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Suppose we are contracting two nodes u, v in G into one node called uv.

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  - Otherwise, x is a red neighbor of u and v
- All other edges are left alone and maintain their color

Twin-width

- Repeatedly applying the contraction operation to nodes of G produces a *contraction sequence* of graphs $G = G_n \to G_{n-1} \to \cdots \to G_2 \to G_1 = K_1$

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- The sequence is a *d-sequence* if the maximum number of red edges in any of the G_i in the contraction sequence is d
- The *twin-width* of G is the minimum d such that there exists a d-sequence of G

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- The number of red edges may increase or decrease

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- If we have the d-sequence for a graph, we can decompose the graph into complete bipartite graphs and do breadth-first-search in $O(n \log n)$ time [BGK$^+$20]
- This works even if the number of edges is $O\left(n^2\right)$

Section 2

Computing Twin-width

It's Hard

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- There are very few practical algorithms for computing the twin-width of a graph in general
- There are some things known for special cases
  - Cographs are the graphs with twin-width zero
  - d-dimensional graphs have twin-width $\leq 3d$ [BKTW20]
  - Planar graphs have twin-width ≤ 9, and bipartite planar graphs have twin-width ≤ 6 [Hli22]
  - Graphs with 4 vertices have twin-width ≤ 1 and graphs with 5 vertices have twin-width ≤ 2 [Das22]
  - In general, bipartite graphs may have arbitrarily large twin-width

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- [SS21] found a way to encode twin-width into a SAT formula
- This is one of the only ways we can reasonable compute twin-widths

Section 3

PACE 2023

A Programming Competition

- **P**arameterized **A**lgorithms and **C**omputational **E**xperiments is a long-term programming competition
- Our goal will be to devise an efficient algorithm to compute the twin-width d of arbitrary graphs and their d-sequence
  - Exact Track: Compute the exact sequence
  - Heuristic Track: Compute an approximate sequence

The Club Submission

- I only found out about this recently so we are a bit behind
- We should make on submission on one track
  - I was thinking about focusing on the exact track
- There are 100 public test cases + 100 test cases
- Score = the number of test cases we can solve in a given time limit
- My goal is just to put together something and see how well we do

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