A Brief Introduction to Knots

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Outline

What is knot? (first attempt)

A **knot** $k : S^1 \to \mathbb{R}^3$ is an **embedding**.

Two knots $k, l : S^1 \to \mathbb{R}^3$ are **equivalent** if there exists an **orientation-perserving homeomorphism** between them.

A knot is called **unknot** if it is homeomorphic to S^1

Defining Terminology

Goal: define enough topology to understand the definitions. Here is the list:

- continuous function
- homeomorphism
- embedding
- orientation

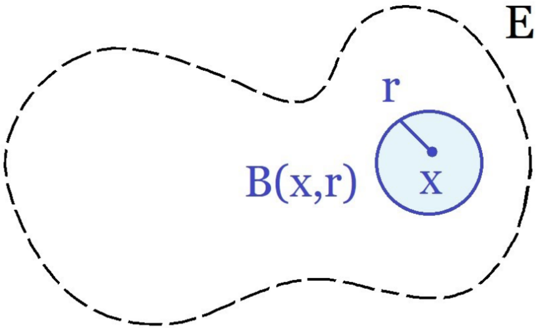
Continuous Function

In topology, the only transformation we are concerned about are the continuous ones.

In analysis, you learned that a function $f : X \to Y$ is continuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Here X and Y are real lines or metric space more generally.

Equivalently, a map $f : X \to Y$ is continuous if for any open set $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in X.

Open Sets



Homeomorphism

A continuous map $f : X \to Y$ is a homeomorphism if it has a continuous inverse $f^{-1} : Y \to X$.

Equivalently, a map $f : X \to Y$ is a homeomorphism iff it is a bijective continuous open map. That means, f is a bijection and

- for any open set $U \subseteq X$, $f(U)$ is open in Y and
- for any open set $V \subseteq Y$, $f^{-1}(U)$ is open in X.

Homeomorphism

Notice that a homeomorphism is not a continuous bijection. Here is an example:

There is a continuous bijection $f : [0, 2\pi) \to S^1$, where

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is the unit sphere.

However, f is not a homeomorphism since there is no continuous inverse. Equivalently, f it is not open, since $f([0, 1))$ is not an open set in S^1.

Embedding

A map $f : X \to Y$ is an embedding if X and $f(X)$ are homeomorphic, i.e. there exists a homeomorphism $g : X \to f(X)$.

Orientation

I will assume the intuitive definition of orientation. A surface is orientable if "clockwise" or "anti-clockwise" direction can be defined.

For example, a Mobiüs strip is not orientable.

Orientation

Here is a formal definition if you wish:

Let S be a surface. S is orientable if its first homology group $H_1(S)$ has a trivial torsion subgroup. That means, $H_1(S)$ is a free abelian group.

S is non-orientable if $H_1(S) = \mathbb{Z}^n + \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}^n$ is the free abelian group of rank n.

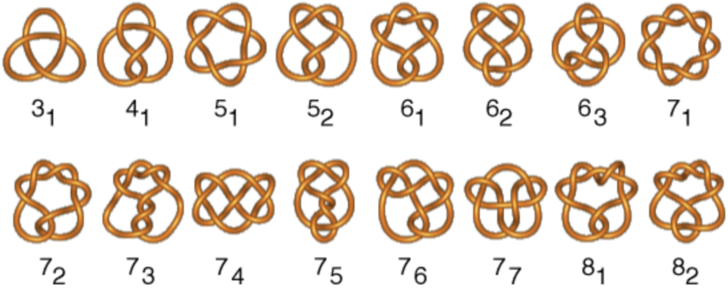
What is knot? (second attempt)

A **knot** $k : S^1 \to \mathbb{R}^3$ is an **embedding**.

Two knots $k, l : S^1 \to \mathbb{R}^3$ are **equivalent** if there exists an **orientation-perserving homeomorphism** between them.

A knot is called **unknot** if it is homeomorphic to S^1

Examples of Knots



Why $\mathbb{R}^3$?

It is unusual for mathematician to restrict our attention to a specific space. Usually, we try to generalize things as much as possible.

There might be two reasons behind to choice of $\mathbb{R}^3$:

- Embeddings in other dimensions $k : S^1 \to \mathbb{R}^n$ $(n \neq 3)$ is too hard and we cannot understand them. So we pick this special case to study first.
- Embeddings in other dimensions are too easy and there is not much to say about them.

Why $\mathbb{R}^3$?

Theorem

Any embedding $f : S^1 \to \mathbb{R}^2$ *is an unknot.*

Proof.

In $\mathbb{R}^2$, the knot cannot possible cross itself, so it is trivially unknotted. □

Why $\mathbb{R}^3$?

Theorem

Any embedding $f : S^1 \to \mathbb{R}^n$ *for* $n \geq 4$ *is an unknot.*

Higher Dimensional Knots

There are much to say about higher dimensional knots, but I will give a definition and a few remarks.

A n-knot is an embedding $f : S^n \to \mathbb{R}^{n+2}$. The most common object of study is the codimension 2 knots, but I will discuss that in a bit.

Higher Dimensional Knots

There are less tools to study knots in higher dimensions (none without introducing much more sophisticed math tools).

There are two common techniques involved:

- surgery theory for geometric properties (genus) about higher dimensional knots
- algebraic techniques classifying knot invariants via homotopy, homology, or cohomology groups of the knot-complement (since knots themselves are always S^n up to homeomorphism).
- differential tools in differentiable categories and PL tools in piece-wise linear categories.
- any combination of the above in the correct category.

Codimension Requirements for Higher Knots

[Zee63] and [Sta63] proved that embeddings $k : S^n \to S^m$ with codimension $m - n \geq 3$ are unknotted in the piecewise linear and topological categories.

There is codimension ≥ 3 knotting in the differentiable category. Differentiable embeddings $k : S^n \to S^m$ with $m - n \geq 3$ were classified by [Hae66] and [Lev65].

Codimension 1 embeddings are almost as interesting as codimension 2 embeddings, although they do not have the intuitive appeal of classical knot theory. Many of the techniques used in codimension 2 make crucial use of codimension 1 embeddings, such as spanning surfaces.

Other Interesting Structure for Knot Theorists

- Many problems about knot can be easier if we understand link. Link is an embedding $l : \coprod_{i=1}^{n} S^1 \to \mathbb{R}^3$ of n copies of S^1.
- Ribbon knot is also interesting, which is an immersion (self-intersection) $r : D^2 \to \mathbb{R}^3$ with only ribbon-singularities.

Connection to Other Subjects

- knot theory as first step of proving Poincaré conjecture: property P conjecture (Dehn surgery on a knot) as a special case for the Poincaré conjecture
- knot theory in chemistry: study the geometric property of non-planar molecures, e.g. examining chirality via knot theory
- knot theory in material science: study the topological structure of materials for their physical/chemical properties, e.g. the materials and topological property of touchscreens

Connection to Other Subjects

- knot theory in quantum computing: redrawing quantum circuits in ZX-calculus (knot diagrams) and implement it to find better quantum algorithm.
- knot theory in quantum gravity: Topological Quantum Field Theory (TQFT)
- knot theory in particle physics: string theory
- complexity theory and knot theory: determining the complexity for determining if a knot is unknot

Papers that I Mentioned I

Andre Haefliger.

Differentiable embeddings of sn in sn+q for $q > 2$.

Annals of Mathematics, 83(3):402–436, 1966.

Jerome Levine.

A classification of differentiable knots.

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John Stallings.

On topologically unknotted spheres.

Annals of Mathematics, 77(3):490–503, 1963.

Sir Erik Christopher Zeeman.

Unknotting combinatorial balls.

Annals of Mathematics, 78(3):501–526, 1963.