Estimates in Analytical Number Theory

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The use of bigO notation in Analytic number theory

- Presumably, You've seen the BigO notation in your CS classes. They are generally used to notify the speed or the size of a program
- For example, BFS takes O(V + E) time and hashing algorithms like maps generally take O(n) space.
- Similar ideas are used in Analytical Number theory as well. The BigO is used to usually indicate the size of the error.
  - It is very useful too, since you can pull out the bigO out of integrals and limits, more useful that other estimates such as smallO. $\int O(\log x)dx = O(\int \log x dx)$

Some examples

The Prime Number Theorem is equivalent to the statement:

$$\pi(x) = \frac{x}{\log(x)} + o(\frac{x}{\log(x)}))$$

Where $\pi(x)$ is the number of prime numbers less than or equal to x.

Knowing

$$Li(x) = \int_2^x \frac{1}{\log(t)} dt$$

the current best known estimate is

$$\pi(x) = Li(x) + O(x \exp\{-(\log(x))^\alpha\}); \alpha \in \mathbf{R}, \alpha < \frac{3}{5}$$

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Finally, the Riemann Hypothesis is equivalent to:

$$\pi(x) = Li(x) + O(x^{\frac{1}{2}+\epsilon}); \epsilon > 0$$

Manipulation with BigO notation

Proof of Sterling Formula

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Theorem

S(N) $= \sum_{n \leq N} \log(n) = N(\log(N) - 1) + \frac{\log(N)}{2} + c + O(\frac{1}{N})$

Proof

1. We will use special case of Euler's Summation Formula here

$$\sum_{n \leq x} f(n) = \int_1^x f(t)dt + \int_1^x \{t\} f'(t)dt - \{x\} f(x) + f(1)$$

   Here $\{x\}$ indicates the fractional part of x, and $f'(x)$ indicates the derivative of f(x). There isn't enough time for a proof, so assume this formula exists.

2. Use Euler's Summation formula but with $f(x) = \log(x)$, we note that $\{N\} = 0, f(1) = 0$. Hence $S(N) = I_1(N) + I_2(N)$

Manipulation with BigO notation

3. $I_1(N) = \int_1^N \log(t)dt = N\log(N) - N + 1$
4. $I_2(N) = \int_1^N \frac{\{t\}}{t}dt = \int_1^N \frac{1}{2t}dt + \int_1^N \frac{\{t\}-\frac{1}{2}}{t}dt = \frac{\log(N)}{2} + I_3(N)$

Combining these formulas gives us:

$$S(N) = N\log(N) - N + 1 - \frac{\log(N)}{2} + I_3(N)$$

Now, we use integration by parts on $I_3(N)$

$$I_3(N) = \left.\frac{R(t)}{t}\right|_1^N + \int_1^N \frac{R(t)}{t^2}dt$$

where $R(N) = \int_1^N \{t\} - \frac{1}{2}dt$

Manipulation with BigO notation

We note the following:

- $\rho(t) = \{t\} - \frac{1}{2}$ is periodic with period 1 and $|\rho(t)| \leq \frac{1}{2}$
- $\int_k^{k+1} \rho(t)dt = 0$ for any integer k, because of the way it's structured.
- Finally, $|R(t)| \leq \frac{1}{2}$ for any t

Hence we have that $I_3(N) = \int_1^N \frac{R(t)}{t^2}dt$. This specific integral now converges as $N \to \infty$, since $\frac{|R(t)|}{t^2} \leq \frac{(\frac{1}{2})}{t^2}$

Therefore

$$I_3(N) = I - \int_N^\infty \frac{R(t)}{t^2}dt = I - O(\int_N^\infty \frac{1}{t^2}dt) = I - O(\frac{1}{N})$$

where $I = \int_1^\infty \frac{R(t)}{t^2}dt$, a constant. $\square$

Manipulation with BigO notation

The end is near

Recapping what we've done so far, we have shown that:

$$S(N) = \sum_{n \leq N} \log(n) = N\log(N) - N + C - \frac{\log(N)}{2} - O(\frac{1}{N})$$

However, we note that $n! = \exp\left\{\sum_{k \leq n} \log(k)\right\}$, exponentiating the formula that we just derived we get:

$$n! = n^n e^{-n}\sqrt{n}e^C(1 + O(\frac{1}{N}))$$

due to the taylor series of exponentials. □

Note: We can accurately get the constant value to be $\sqrt{2\pi}$, but that is much more difficult, and will require a course in number theory

Questions?