Random Walks and Markov Chains

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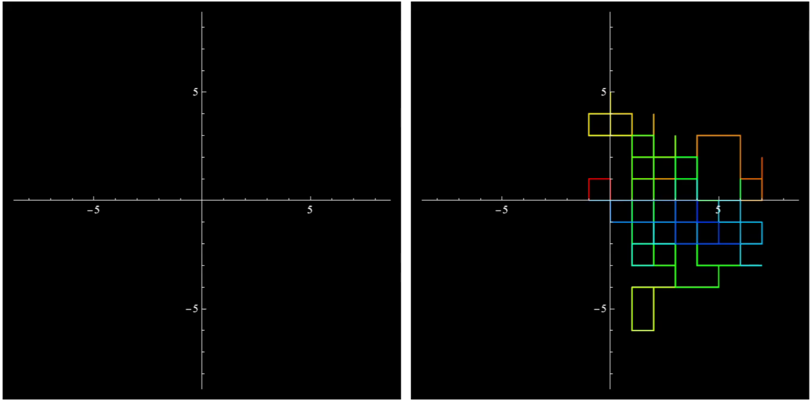
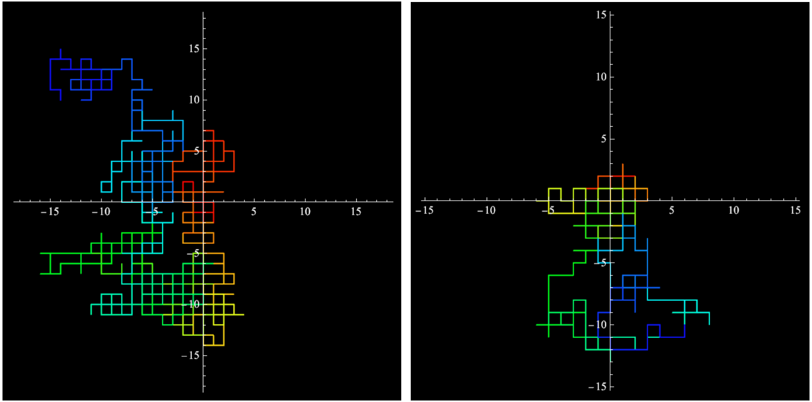
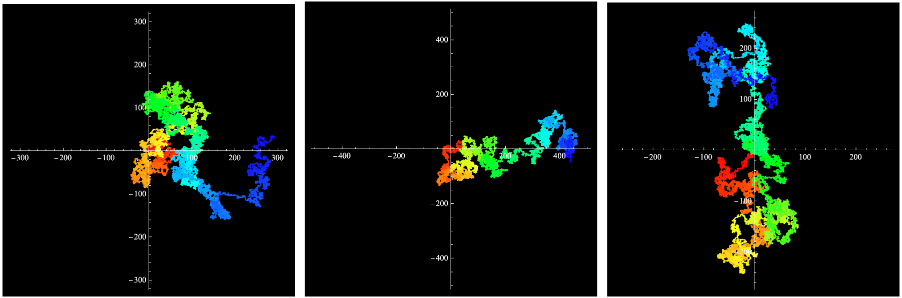
Random Walks



More Random Walks



More Random Walks



Section 1

Markov Chains

Definitions

- A *stochastic process* is a sequence of random variables $[X_0, X_1, X_2, X_3, \ldots]$ where $X_i \in \mathbb{Z}$.
- If $X_t = i$, we say the process has value i at time t.
- A *Markov Chain* is a random process with the Markov property:

$$\mathbb{P}(X_{t+1} = i_{t+1} \mid X_t = i_t, X_{t-1} = i_{t-1}, \ldots, X_0 = i_0)$$
$$= \mathbb{P}(X_{t+1} = i_{t+1} \mid X_t = i_t)$$

Transition Probabilities

- By the Markov property, the transition probabilities between any two states can be expressed as a single value:

$$p_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i)$$

Theorem

A random process $(X_t)_{t \geq 0}$ is Markov if and only if $\forall i_0, \ldots, i_n$:

$$\mathbb{P}(X_1 = i_1, \ldots, X_n = i_n \mid X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-1} i_n}$$

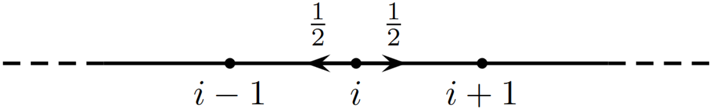
Transition Matrix

The *transition matrix* of a Markov Chain contains all transition probabilities:

$$\begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example: Random Walk on Number Line

Consider a symmetric random walk on the number line:



$$\begin{pmatrix} & & \dots & \dots & & \\ & 0 & 1/2 & 0 & 0 & \\ \dots & 1/2 & 0 & 1/2 & 0 & \dots \\ \dots & 0 & 1/2 & 0 & 1/2 & \dots \\ & 0 & 0 & 1/2 & 0 & \\ & & \dots & \dots & & \end{pmatrix}$$

Recurrence and Transience

- We say state i is *recurrent* if, starting at i,

$$\mathbb{P}(\text{chain visits } i \text{ infinitely many times}) = 1$$

- We say state i is *transient* if, starting at i,

$$\mathbb{P}(\text{chain visits } i \text{ infinitely many times}) = 0$$

https://illinois.zoom.us/j/84778016612?pwd=b0FDVWwxdDlMMjR4TzNqamo0ZGZK

Ex. In the chain $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, state 0 is transient, state 1 is recurrent.

Recurrence and Transience

Theorem

For a Markov Chain starting at i,

1. $\mathbb{P}(\text{chain eventually returns to } i) = 1 \Leftrightarrow$
$$i \text{ is recurrent } \& \ \mathbb{E}[\text{number of visits to } i] = \infty,$$
2. $\mathbb{P}(\text{chain eventually returns to } i) < 1 \Leftrightarrow$
$$i \text{ is transient } \& \ \mathbb{E}[\text{number of visits to } i] < \infty.$$

Goal: prove $(0,0)$ is a recurrent state in a random walk on $\mathbb{Z}^2$, and $(0,0,0)$ is a transient state in a random walk on $\mathbb{Z}^3$.

Section 2

A Proof

Recurrence of 2-D Walk

Want to show: $\mathbb{E}[\text{number of visits to } (0{,}0)] = \infty$.

Proof

1. Note that $\mathbb{E}[\text{number of visits to } (0{,}0)] = \sum_{t=0}^{\infty} p_{00}^{(t)}$;
2. Since returning to the origin can only be accomplished with an even number of steps, we will only consider the even terms.
3. Then, $p_{00}^{2t} = \frac{1}{4^t} \sum_{i=0}^{t} \frac{(2t)!}{i!i!(t-i)!(t-i)!} = \frac{1}{4^{2t}} \binom{2t}{t} \sum_{i=0}^{t} \frac{n!n!}{i!i!(t-i)!(t-i)!} = \frac{1}{4^{2t}} \binom{2t}{t} \sum_{i=0}^{t} \binom{t}{i}^2 = \frac{1}{4^{2t}} \binom{2t}{t}^2 \sim \frac{1}{\pi t}$ by Stirling's approximation.
4. Since $\sum p_{00}^{(t)} \sim \sum \frac{1}{\pi t}$, we may conclude that both series diverge, hence (0,0) is recurrent.

Transience of 3-D Walk

Want to show: $\mathbb{E}[\text{number of visits to } (0,0,0)] < \infty$.

Proof

1. Note that $\mathbb{E}[\text{number of visits to } (0,0,0)] = \sum_{t=0}^{\infty} p_{00}^{(t)}$;
2. Since returning to the origin can only be accomplished with an even number of steps, we will only consider the even terms.
3. Then, $p_{00}^{2t} = \frac{1}{6^{2t}} \sum_{i+j+k=t} \frac{(2t)!}{(i!j!k!)^2} = \frac{1}{6^{2t}} \binom{2t}{t} \sum_{i=0}^{t} \binom{t}{i,j,k}^2 \leq \frac{1}{6^{2t}} \binom{2n}{n} \sum_{i=0}^{t} \binom{n}{n/3,n/3,n/3}^2 = \frac{1}{2^{2t}} \binom{2n}{n} \binom{n}{n/3,n/3,n/3} \frac{1}{3^t} \sim \frac{1}{2} \left(\frac{3}{\pi t}\right)^{3/2}$
4. Since $\sum p_{00}^{(t)} \leq \sum \frac{1}{2} \left(\frac{3}{\pi t}\right)^{3/2}$, we may conclude that both series converge, hence (0,0,0) is recurrent.

Questions?

Acknowledgement

The material of this talk was adapted from MATH466 taught by Prof. Renming Song, with the proof taken from [Nor97].

A drunk man will find his way home, but a drunk bird may get lost forever.

— Shizuo Kakutani

Bibliography I

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Markov Chains.

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