Fibonacci Heaps

Porter Shawver

Outline

Section 1

Priority Queue Background

Priority Queues

For the entirety of this presentation, I will be using minimum priority queues, but maximum priority queues are implemented the same way.

Operations:

- Insert - add an element to the queue.
- FindMin - return the smallest element.
- DeleteMin - remove the smallest element.
- DecreaseKey - decrease an element.

Binary Heap

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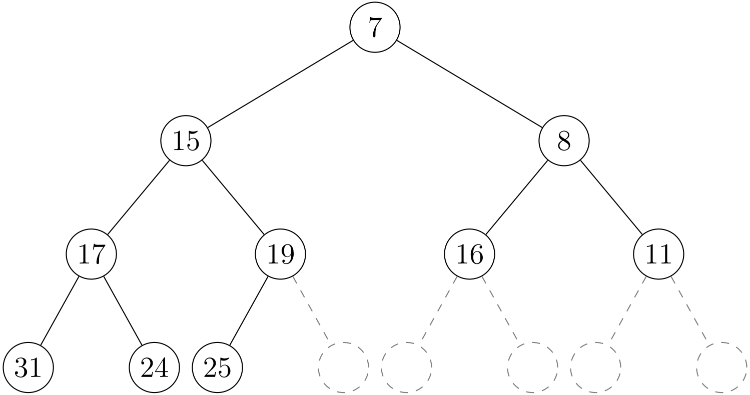
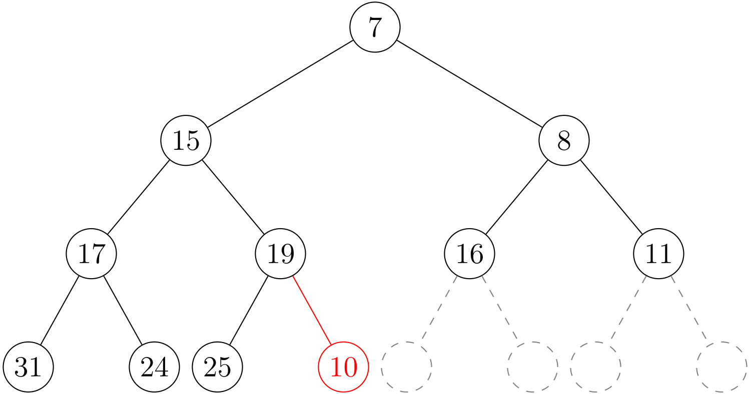
1. All children must have a value greater than their parent ("heap property"). Thus, the root should be the next value out of the queue.
2. The tree is complete (every level must be completely full, except the last). This allows us to use an array, and helps with worst case runtime.

Binary Heap Example



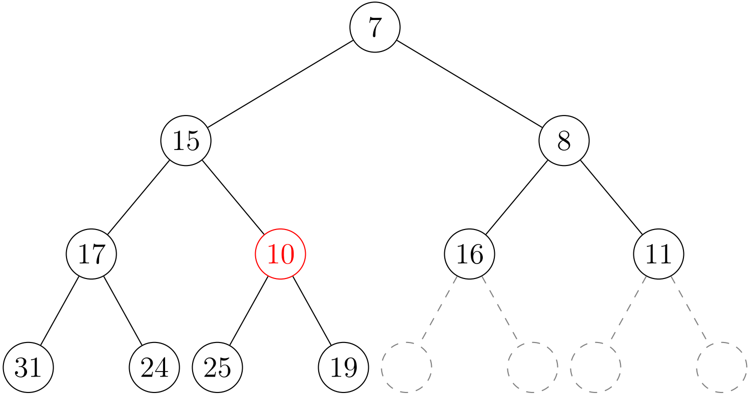
Binary Heap INSERT

1. Add value to first open space.



Binary Heap INSERT

2. Swap with parent until the parent is smaller to maintain heap property.



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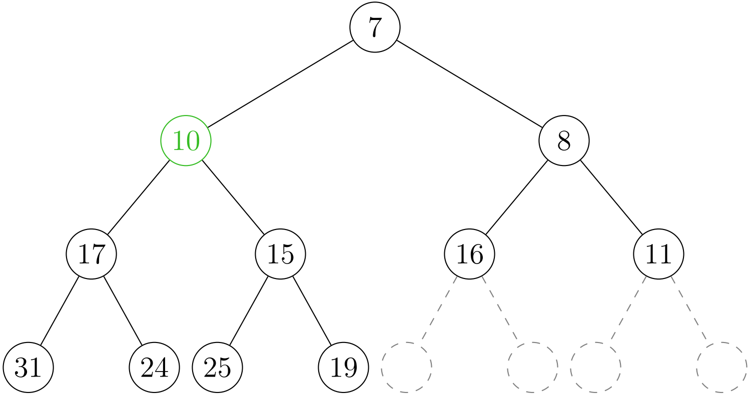


Binary Heap INSERT

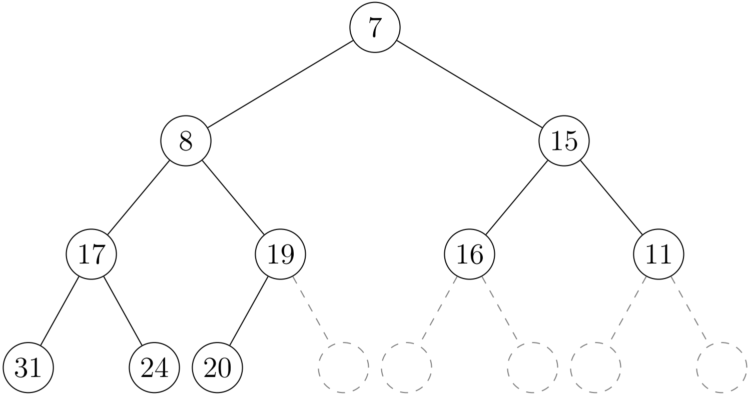
Worst case, the inserted value is the new smallest value, and must be swapped all the way to the top: $O(\log n)$.

Binary Heap FindMin

The root is always the min, so finding it is $O(1)$.

Binary Heap DeleteMin

1. Return the root value, and replace the root node with the last element in the array.



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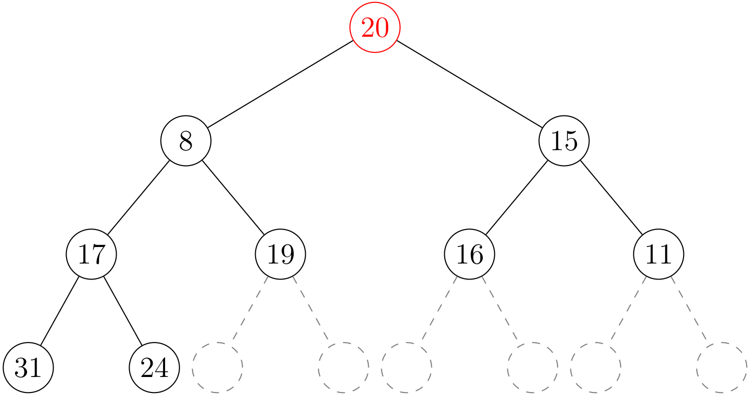
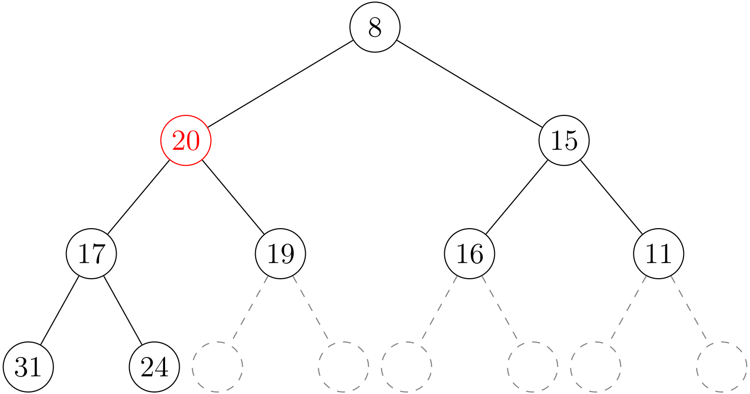


Binary Heap DeleteMin

2. Swap with smallest child until both children are larger.



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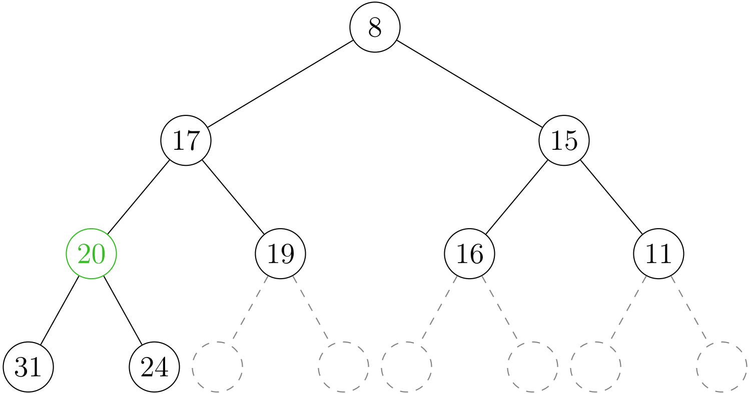

Binary Heap DeleteMin

Worst case, the replacement value will be swapped all the way to the bottom. Because this is a binary tree, we know this is $O(\log n)$.

Binary Heap DecreaseKey

This is the same as insert: decrease the priority of the node, then swap it with its parent until the parent is larger: $O(\log n)$.

Binary vs. Fibonacci Heap Runtimes

	Binary Heap	Fibonacci Heap
Insert	$O(\log n)$	$O(1)$
FindMin	$O(1)$	$O(1)$
DeleteMin	$O(\log n)$	$O(\log n)$
DecreaseKey	$O(\log n)$	$O(1)$

Questions?

Section 2

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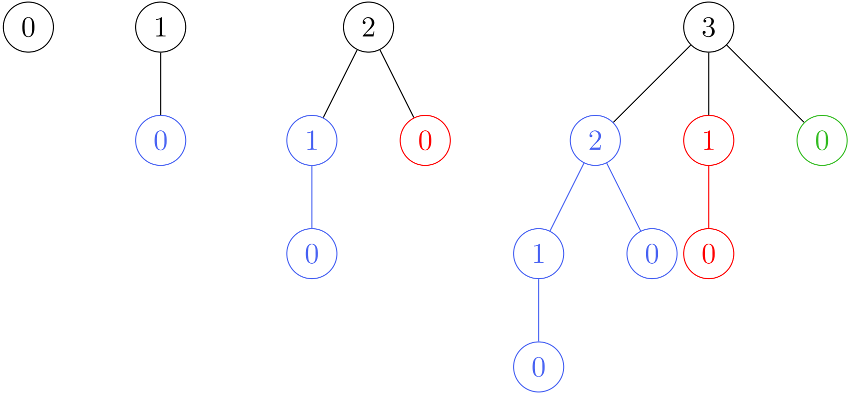
- Level ℓ has $\binom{k}{\ell}$ nodes in a binomial tree of order k.
- Thus, a tree of order k has 2^k nodes.
- Thus, the largest degree is bounded by $O(\log n)$.

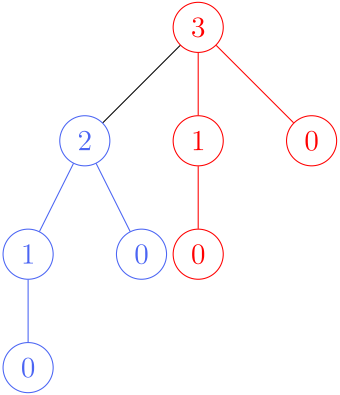
Binomial Heap Example

(All one heap, nodes labeled with their degree.)



Binomial Heap Notes

- Combining two binomial heaps of degree d gives a binomial heap of degree $d + 1$:



Binomial Heap Insert

Using amortized analysis, a binomial heap can insert in $O(1)$ by adding a single node as a new tree.

(More on amortized analysis in a moment.)

Section 3

Fibonacci Heaps

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Very loose structure; could just be a list of all root nodes. We use amortized analysis to get the runtimes we want.

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- Formally, we define $\Delta\phi_o$ as the change in potential after operation o. Then the amortized time is calculated as

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- When we “cash out”, the amortized time is smaller than the actual time, and ϕ decreases (usually to 0).

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- This looks really arbitrary, but we will soon see why it works.

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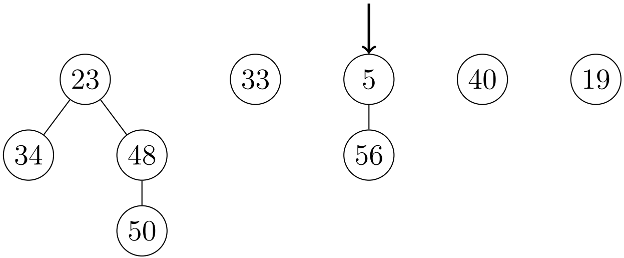
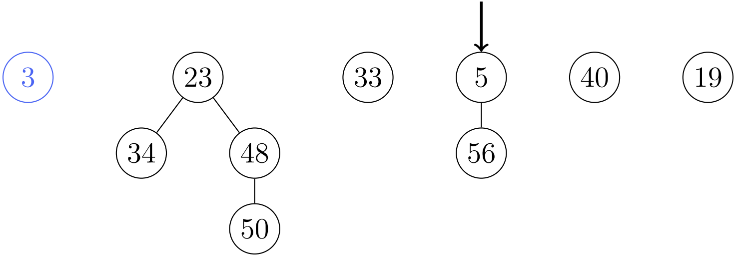
- We maintain a pointer to the minimum element, so if this is the new minimum, update the pointer.
- $O(1)$.

Fibonacci Heap Insert Example



Fibonacci Heap INSERT Example



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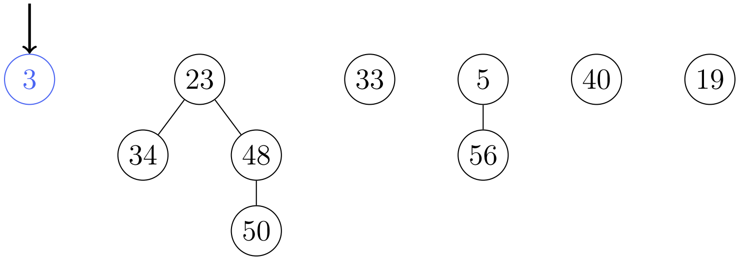

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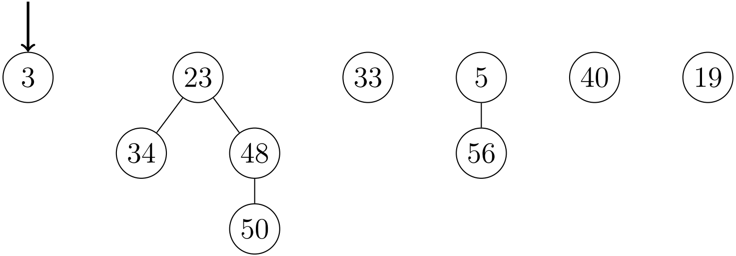
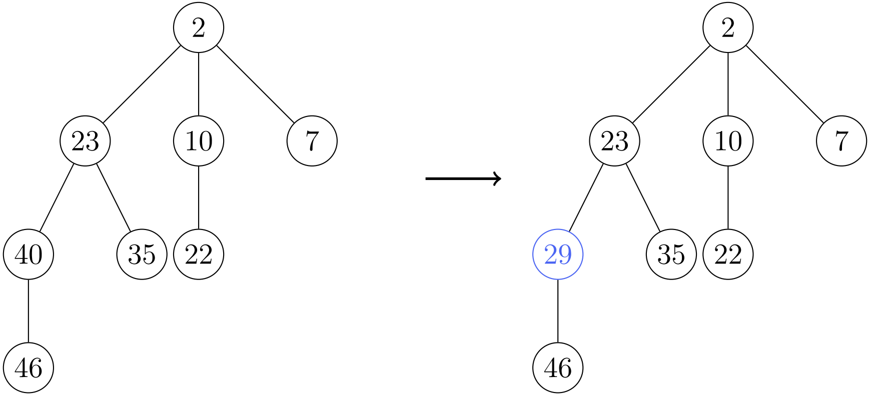


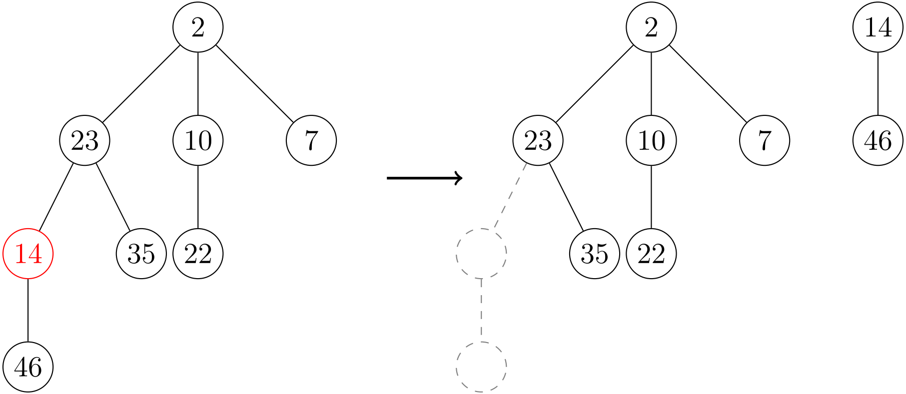
Fibonacci Heap DecreaseKey

1. Decrease the key. If the new value maintains the heap property, we are done.



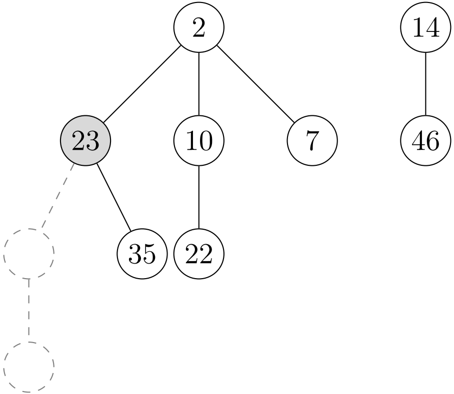
Fibonacci Heap DecreaseKey

2. Otherwise, cut the decreased key (and its subtree) out into a new tree.



Fibonacci Heap DecreaseKey

3. If the parent was unmarked, mark it. Otherwise, cut it out, unmark it, and repeat for this node's parent.



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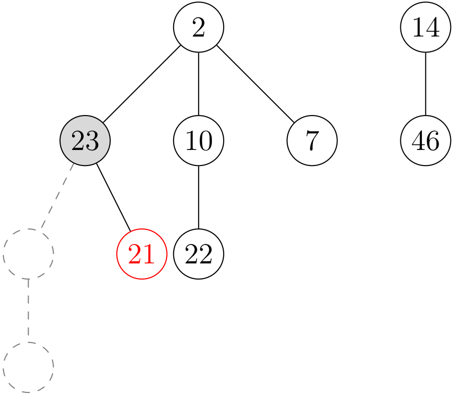
Now, if we cut out 35, we would also cut out 23:



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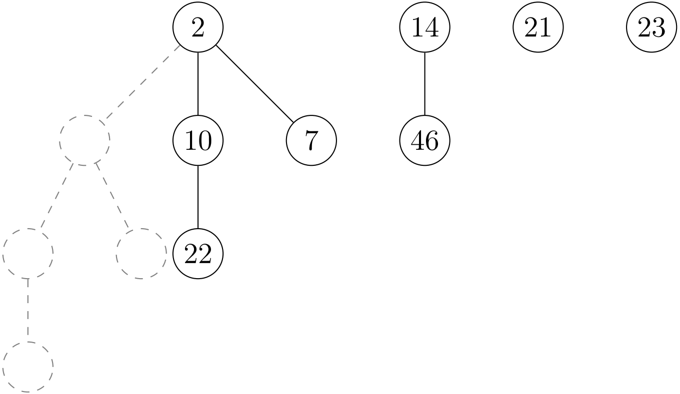

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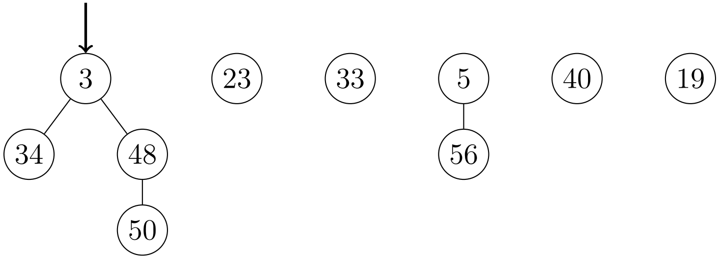
- So our amortized time is $O(1)$.
- Notice our choice of ϕ was important here.

Fibonacci Heap DeleteMin Phase 1

Cut out the minimum node (which must be a root) and make its children roots of new trees.



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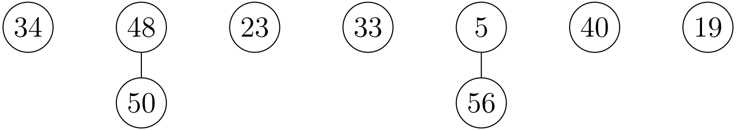

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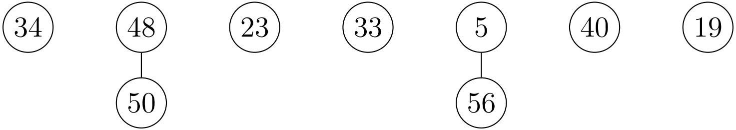

With d children, this operation is $O(d)$. We will see later that $O(d) = O(\log n)$ for a fibonacci heap with n nodes.

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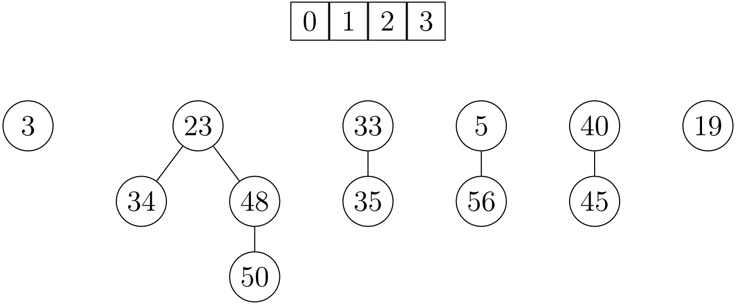
- However, if we "clean up," future calls to DeleteMin could be faster, and could give us a $\Delta\phi$ that helps in amortized analysis.
- Like the last slide, I will assume the largest degree of any root is $O(\log n)$, and prove this later.

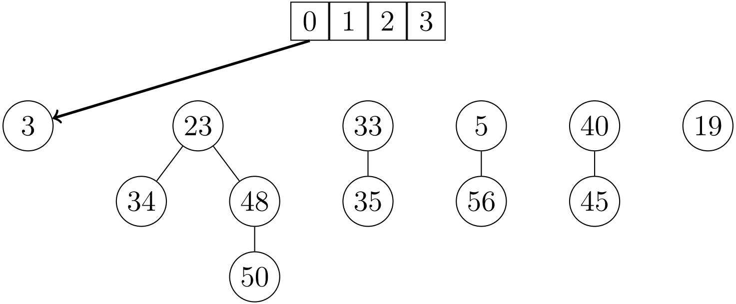
Fibonacci Heap DeleteMin Phase 2

1. Create an array of length $O(\log n)$ to track trees of every possible degree.



Fibonacci Heap DELETEMIN Phase 2

2. Put each tree into an index of the array. If there is already a tree there, merge them to create a tree of degree +1.



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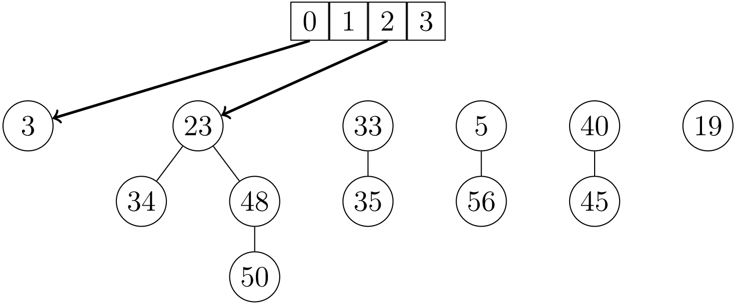

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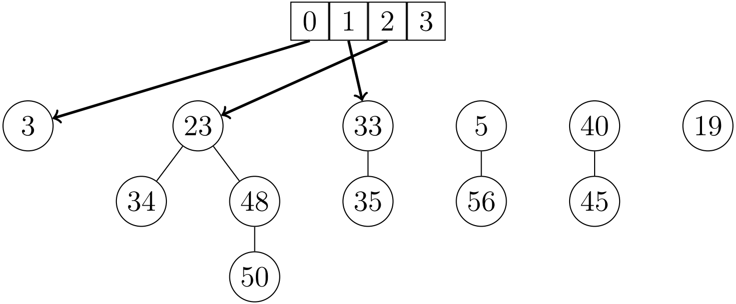

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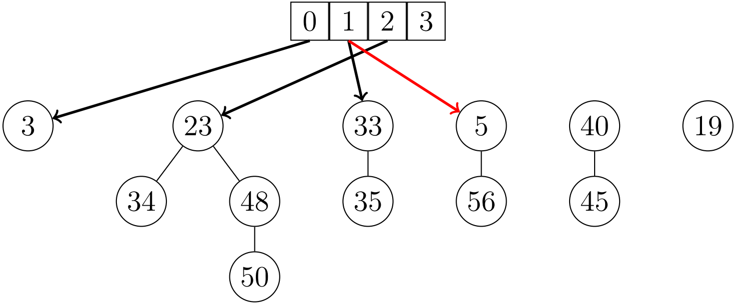

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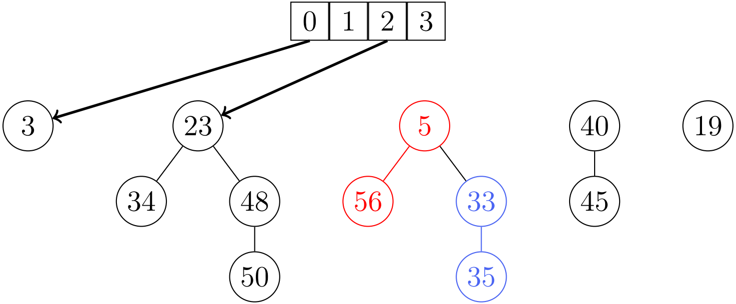

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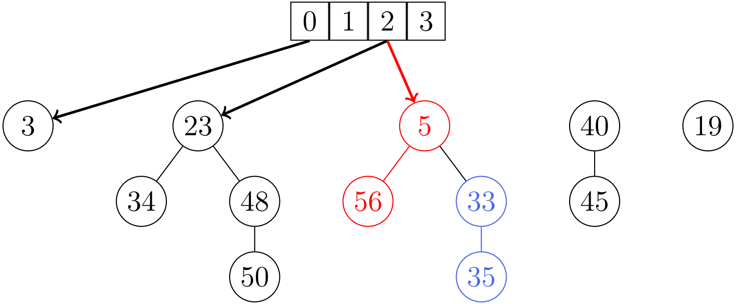

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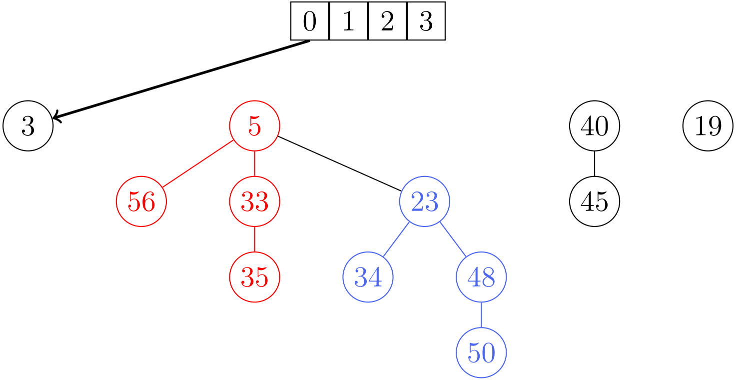

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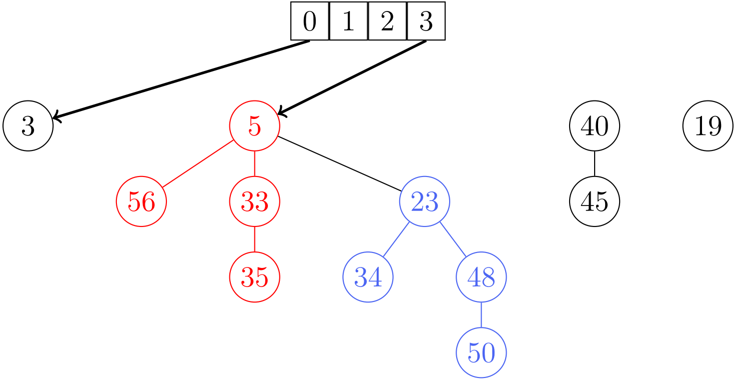

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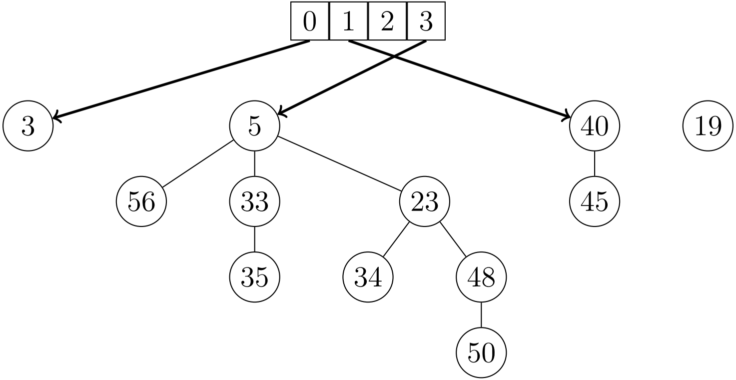

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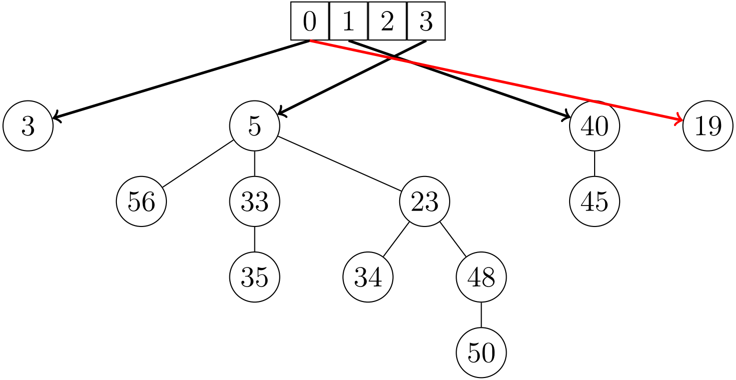

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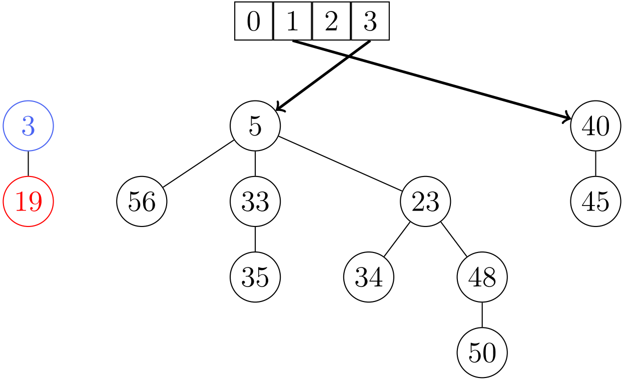

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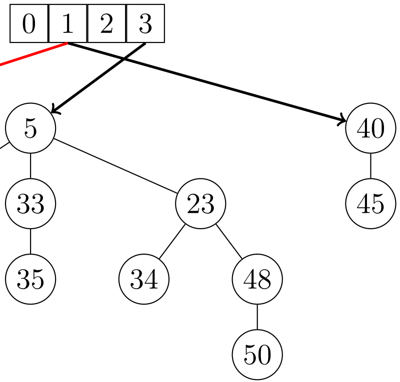

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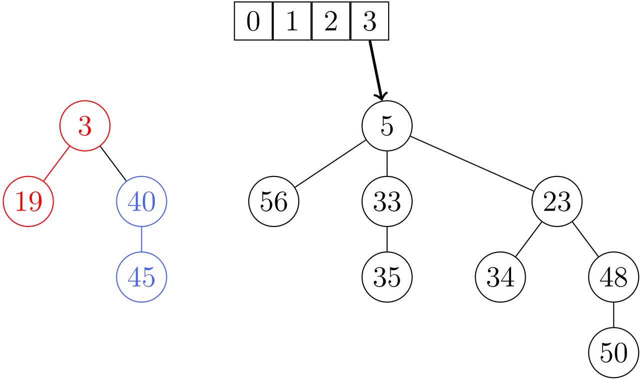

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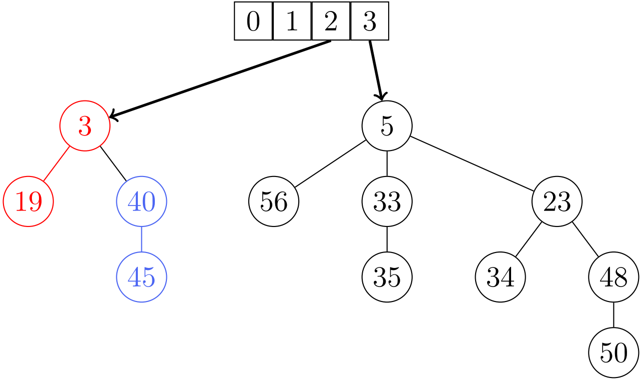

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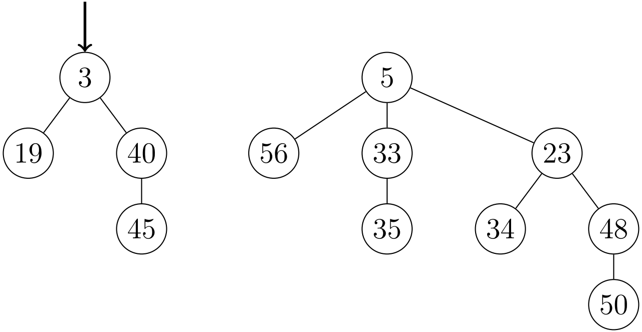
Fibonacci Heap DELETEMIN Phase 3

Find the minimum of the $O(\log n)$ roots that remain in $O(\log n)$ time.



Fibonacci Heap DeleteMin Recap

Of all 3 phases, the slowest runtime was $O(\log\ n)$, so the overall runtime is $O(\log n)$.

Questions?

Fibonacci Heap Degree Bound Proof

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- Thus, we need to show that n grows exponentially with respect to d.

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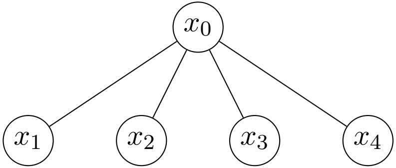

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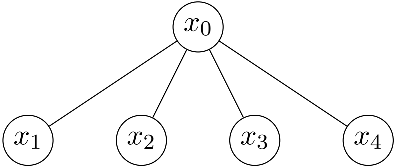

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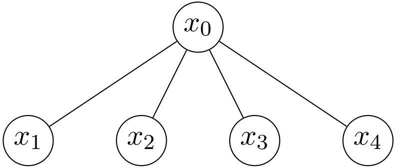

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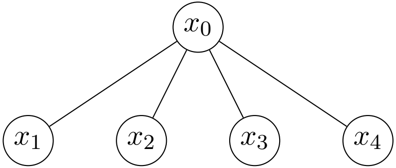

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- In general, the i^{th} child must have degree at least $i - 2$.

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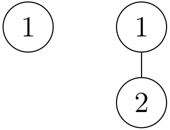
- Minimum for degrees 0 and 1 (base cases) are trivial:



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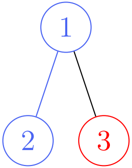

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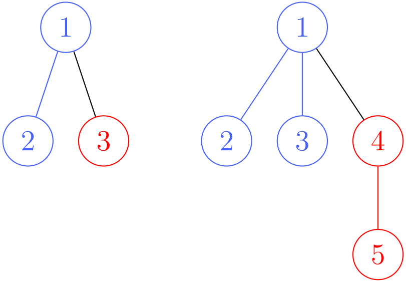

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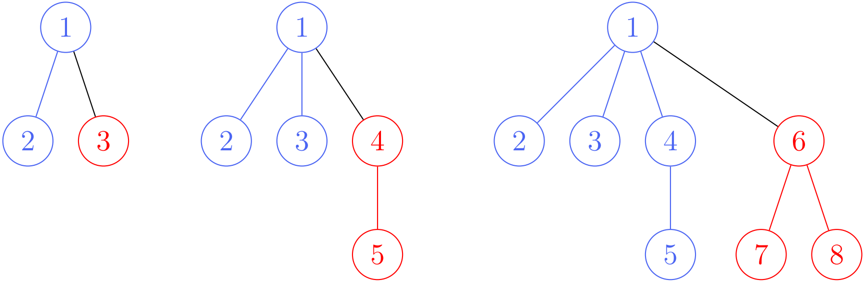

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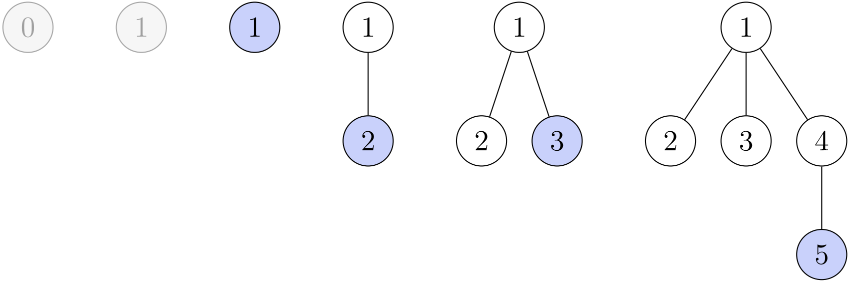

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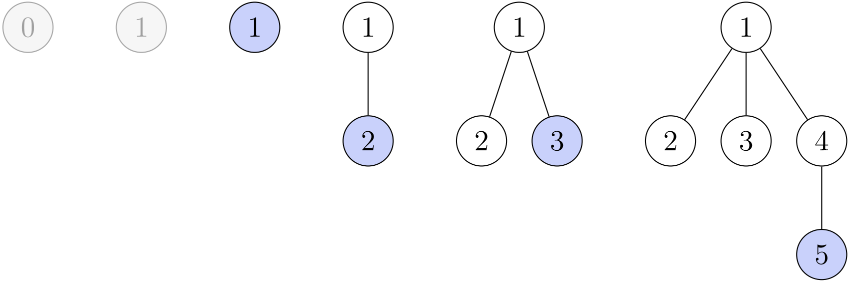

- We know that the growth of Fibonacci approaches $\Phi \approx 1.618$, so $n \geq \Phi^d$. which satisfies our goal.

Recap

	Binary Heap	Binomial Heap	Fibonacci Heap
Insert	$O(\log n)$	$O(1)$	$O(1)$
FindMin	$O(1)$	$O(1)$	$O(1)$
DeleteMin	$O(\log n)$	$O(\log n)$	$O(\log n)$
DecreaseKey	$O(\log n)$	$O(\log n)$	$O(1)$

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- This leads to large big-O constants and heavy reliance on pointers that gives much worse cache performance than binary heaps.
- So, Fibonacci heaps are not often used, because they are too slow in practice.
- Fibonacci heaps can still be faster for really large amounts of data.

Questions?

Computers are useless. They only give you the answers.

— Pablo Picasso (1979)

Bibliography I

Wikimedia Foundation. (2024, January 17). *Fibonacci heap*. Wikipedia. https://en.wikipedia.org/wiki/Fibonacci_heap