Dynamics and Chaos: the Logistic Map

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Section 1

Introduction: $0 < \mu < 3$

Definition

A 1D map can be expressed in the form of $x_{n+1} = f(x_n)$.

Definition

The **logistic map** is a family of functions
$f\mu : \mathbb{R} \to \mathbb{R}, x \mapsto \mu x(1-x), \mu \in \mathbb{R}$.

Specifically, we will investigate $\mu > 0$.

Immediate Observations

- Recall that fixed point x_* is **stable/attracting** if $|f'(x_*)| < 1$ and **unstable/repelling** if $|f'(x_*)| > 1$. Bifurcations usually occur at the **marginally stable** point of $|f'(x_*)| = 1$.
- The initial fixed points for small μ are:
$x = \mu x(1-x) \rightarrow x_* = 0, 1 - 1/\mu$. $f'_\mu = \mu(1-2x)$, so $x_* = 0$ is stable for $0 < \mu < 1$, and $x_* = 1 - 1/\mu$ is stable for $1 < \mu < 3$.

The $1 < \mu < 3$ Case

Proposition

Suppose $\mu > 1$. If $x < 0$ or $x > 1$, then $\lim_{n\to\infty} f_\mu^n(x) = -\infty$.

Proof

Note that if $x > 1$, $f_\mu(x) = \mu x(1 - x) < 0$, reducing to the $x < 0$ case. Note that $x > f_\mu(x)$, so $(f_\mu^n(x))_n$ forms a decreasing sequence. We now show that this sequence cannot converge. Suppose for some $x < 0$, $f_\mu^n(x) \to p$ for some p. Then $f_\mu^{n+1}(x) \to p$. But by continuity of f, $f_\mu^{n+1}(x) \to f_\mu(p) < p$. Contradiction.

The $1 < \mu < 3$ Case (cont.)

Proposition

If $0 < x < 1$, then $\lim_{n\to\infty} f_\mu^n(x) = x_*$, where $x_* = 1 - 1/\mu$ (as expected, because x_* is an attracting fixed point.

Proof (draw a picture)

We split into 3 cases: $1 < \mu < 2$, $\mu = 2$, $2 < \mu < 3$. One can check for the second case explicitly. For $1 < \mu < 2$, $x_* < 1/2$, and $f_\mu(x) < x$ for $0 < x < 1/2$. Thus, for $0 < x < x_*$, $f_\mu(x) > x$, and $(f_\mu^n(x))_n$ converges to x_*. Similarly, for $x_* < x < 1/2$, $f_\mu(x) < x$, and $(f_\mu^n(x))_n$ converges to x_*. Note $f((1/2, 1)) = (0, 1/2)$, and $1/2 < x < 1$ is proved as well. For $2 < \mu < 3$, $x_* > 1/2$. Define $x'_* = 1/\mu$ (reflecting x_* across $x = 1/2$). For $0 < x < x'_*$, $f_\mu(x) > x$. For $x'_* < x < x_*$, although $f_\mu(x) > x_*$, note that $x < f_\mu^2(x) < x_*$ (can be shown by taking derivative). Thus, we have proved for $0 < x < x_*$. For $x_* < x < 1$, note that $f((x_*, 1)) = (0, x_*)$, hence proved.

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The $0 < \mu < 1$ Case

- We have similar results.
- Now $x_* = 1 - 1/\mu < 0$.
- Specifically, for $x < x_*$ or $x > 1/\mu$, $\lim_{n\to\infty} f_\mu^n(x) = -\infty$, and for $x_* < x < 1/\mu$, $\lim_{n\to\infty} f_\mu^n(x) = 0$, as expected, because 0 is an attracting fixed point.

For $\mu > 3$...

- Period doubling for $\mu < 3.56995$.
- Most values of $\mu \in (3.56995, 4)$ exhibit chaotic behavior, but there are still certain isolated ranges of μ that show non-chaotic behavior; these are sometimes called islands of stability (Wikipedia).
- All $\mu > 4$ exhibit chaotic behavior.

Section 2

Bifurcations: $3 < \mu < \mu_\infty$

Bifurcation

- Let λ be a external parameter. Let $f : X \to X$ be a function dependent on λ. Varying λ generates a family of functions, denoted f_λ, where each function in the family uses a different value of λ.
- As we change f by varying λ, there are certain points in the family where the qualitative behavior of the function changes. These changes are called **bifurcations**, and the values of the parameter λ where these changes occur are called **bifurcation points**.

Period Doubling Bifurcation: Definition

Definition

A family of functions f_λ undergoes a **period doubling bifurcation** at $\lambda = \lambda_0$ if $\exists$ open interval I, $\epsilon > 0$ s.t.:

1. $\forall \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon], \exists!$ fixed point $p_\lambda \in I$ for f_λ.
2. $\forall \lambda \in (\lambda_0 - \epsilon, \lambda_0], f_\lambda$ has no cycles of period 2 in I and p_λ is attracting (resp. repelling).
3. $\forall \lambda \in (\lambda_0, \lambda_0 + \epsilon), \exists!$ 2-cycle q_λ^1, q_λ^2 in I that is attracting (resp. repelling). Meantime, fixed point p_λ is repelling (resp. attracting).
4. As $\lambda \to \lambda_0$, $q_\lambda^i \to p_{\lambda_0}$.

First Period Doubling Bifurcation

- At $\mu = 3$, $x_* = 1 - 1/3 = 2/3$, and $f'_\mu(x_*) = \mu(1 - 2x_*) = 3(1 - 2 \cdot 2/3) = -1$, which is a marginally stable point, causing the first bifurcation.
- The 2 periodic points (denoted x_0 and x_1) after the bifurcation must both satisfy $x = f_\mu(f_\mu(x)) = f_\mu^2(x)$, which is a quartic equation. Note that 2 roots are the initial fixed points $x_* = 0, 1 - 1/\mu$, so factor them out to get $x_1, x_2 = \frac{1}{2\mu}(\mu + 1 \pm \sqrt{(\mu + 1)(\mu - 3)})$.

Subsequent Period Doubling Bifurcations

- To find the value of μ at which a bifurcation occurs (going from period-2^{n-1} to period-2^n, we use $(f_\mu^{2^{n-1}})'(x_i) = -1$ for all fixed points x_i of $f_\mu^{2^{n-1}}$.
- Using the chain rule,
$$(f_\mu^{2^{n-1}})'(x_0) = f'_\mu(f_\mu^{2^{n-1}-1}(x_0))f'_\mu(f_\mu^{2^{n-1}-2}(x_0))\cdots f'_\mu(f_\mu(x_0))f'_\mu(x_0)$$
$$= f'_\mu(x_{2^{n-1}-1})f'_\mu(x_{2^{n-2}-2})\cdots f'_\mu(x_1)f'_\mu(x_0) = \prod_{i=0}^{2^{n-1}-1} f'_\mu(x_i) =$$
$$\prod_{i=0}^{2^{n-1}-1} \mu(1-2x_i) = -1.$$
- We know where the fixed points x_i are for period-2^{n-1}, so we can solve the equation as a function in μ.

Feigenbaum Constant

- It turns out that if we take the ratio of the space between consecutive period doubling bifurcation points of the logistic map, we approach a value known as the Feigenbaum constant $\delta \approx 4.669...$ (A006890 in OEIS).
- Let q_n be the value of q of the n-th bifurcation. Let $\delta_n := \frac{q_{n+1}-q_n}{q_{n+2}-q_{n+1}}$. Then $\delta := \lim_{n\to\infty} \delta_n$.
- As expected from geometric series, q_n converges. We denote the limit as $\mu_\infty \approx 3.56994...$ (A098587 in OEIS).
- This constant is *universal* for all 1D maps with a single locally quadratic maximum.

Renormalization

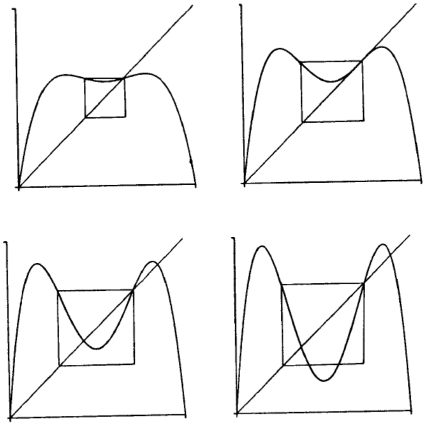

Figure: $f_\mu^2(x)$ for increasing μ, $2 < \mu < 4$.

Renormalization (cont.)

Definition

Let p_μ be the periodic point of period 2 of f_μ, and let p_μ^* be the point s.t. $f(p_\mu^*) = p_\mu$. $L_\mu(x) := \frac{1}{p_\mu^* - p_\mu}(x - p_\mu)$. The **renormalization** of f_μ is defined as $(Rf_\mu)(x) := L_\mu \circ f_\mu^2 \circ L^{-1}(x)$.

- Rf_μ provides a magnified view of the local behavior.
- Observation: if x is a periodic point of period 2 of f_μ, then $L(x)$ is a fixed point of $(Rf_\mu)(x)$.
- Repeated renormalization leads to a sequence of bifurcation points that we expected.

Section 3

Symbolic Dynamics

An Alternative Metric Space

Definition

Let $\Sigma_2 := \{s = (s_0 s_1 \cdots) : s_j = 0 \text{ or } 1 \forall j\}$ be the set of all bitstrings of infinite length. Let $\forall s, t \in \Sigma_2, d(s,t) := \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$.

Proposition

d is a metric on Σ_2.

Observation: $s, t \in \Sigma_2$. If $s_i = t_i \forall i, 0 \leq i \leq n$, then $d(s,t) \leq 2^{-n}$. If $d(s,t) < 2^{-n}$, then $s_i = t_i \forall i, 0 \leq i \leq n$.

The Shift Map σ

Definition

$\sigma : \Sigma_2 \to \Sigma_2, \sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$.

Proposition

σ continuous.

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Properties of σ

Proposition

1. Number of periodic points of period n is 2^n.
2. Periodic points are dense in Σ_2.
3. There exists dense orbit in Σ_2.

Corollary

σ is chaotic in Σ_2.

Proof of Proposition

Proof

1. A periodic point of period n is in the form of $s_0 \cdots s_{n-1} s_0 \cdots s_{n-1} \cdots$, so there are 2^n such periodic points.
2. We want $\forall t \in \Sigma_2, \epsilon > 0 \exists$ periodic point $s : d(s, t) < \epsilon$. Note $\exists M : s_i = t_i \forall 0 \leq i \leq M \to d(s, t) < \epsilon$. Take $s = (t_0 \cdots t_M t_0 \cdots t_M \cdots)$.
3. We want $\exists s \in \Sigma_2 : \forall t \in \Sigma_2, \epsilon > 0 \exists N : d(\sigma^N(s), t) < \epsilon$. Note again $\forall \epsilon \exists M : \sigma^N(s) = t_i \forall 0 \leq i \leq M \to d(\sigma^N(s), t) < \epsilon$. Denote s_M to be a concatenation of an enumeration of all bitstrings of length M. Take $s =$ concatenation of all $s_M \forall M \in \mathbb{N}$.

Section 4

Topological Conjugacy: $\mu > 4$

The Set Λ

Definition

Let $A_n = \{x \in [0,1] : f_\mu^n(x) \in [0,1], f_\mu^{n+1}(x) \notin [0,1]\}$. Define $\Lambda := [0,1] - \bigcup_{n=0}^{\infty} A_n$.

Definition

A set S is a **Cantor set** if it is closed, totally disconnected (contains no intervals as connected subsets), and perfect (every point in S is an accumulation point of all other points in S).

Proposition

Λ is a Cantor set for $\mu > 4$.

Proof of Cantor-ness

Proof

- Closed: we start with a closed interval. At every iteration, an open interval is removed (equivalent of taking the intersection with a closed interval). Intersection of closed intervals are closed.
- Totally disconnected: we shall see this shortly.
- Perfect: suppose not. Then there exists isolated point $p \in \Lambda$. Then exists some i s.t. $\forall x \in \overline{B^o_\epsilon}(p)$, $f^i_\mu(x) < 0$. Thus, f^i_μ is maximized at p, so $(f^i_\mu)'(p) = 0$. This implies that $f'(f^j_\mu(p)) = 0$ for some $j < i$. But this means $f^j_\mu(p) = 1/2$, which is a contradiction because $f^{j+1}_\mu(p) \notin \Lambda$.

Definitions

Definition

A function $f : X \to Y$ between two spaces X, Y is a **homeomorphism** if

1. is a bijection
2. is continuous
3. has continuous inverse

Definition

Let $f : X \to X$, $g : Y \to Y$. Then f, g are **topologically conjugate** if there exists homeomorphism $h : X \to Y$ s.t. $h \circ f = g \circ h$.

A Preliminary Theorem

Definition

Let $S : \Lambda \to \Sigma_2$ be the following function: $x \in \Lambda, S(x) = s_0 s_1 \cdots$, where $s_i = 0$ if $f_\mu^i(x) \in I_0$ and $s_i = 1$ if $f_\mu^i(x) \in I_1$, where $I_0 = \{x \in [0, 1/2] : f_\mu(x) \leq 1\}$ and $I_1 = \{x \in [1/2, 1] : f_\mu(x) \leq 1\}$.

Theorem

If $\mu > 2 + \sqrt{5}$, then $S : \Lambda \to \Sigma_2$ is a homeomorphism.

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Continuity of S

Proof

We will use the sequential criterion of continuity. Consider arbitrary sequence $(x_n) \subset \Lambda$ with $x_n \to x \in \Lambda$. We want $S(x_n) \to S(x) \in \Sigma_2$, i.e., $\forall \epsilon > 0, \exists N : n > N \to d(S(x_n), S(x)) < \epsilon$. Note that $d(S(x_n), S(x)) < \epsilon \leftrightarrow \exists M : (S(x_n))_i = (S(x))_i \forall i \leq M$ (take M s.t. $2^{-M} < \epsilon$). This means that $\forall i \leq M$, $f_\mu^i(x_n)$ and $f_\mu^i(x)$ agree on whether they are in I_0 or I_1. Let J_i be the interval (either I_0 or I_1) that $f_\mu^i(x)$ is in. For each $i, 0 \leq i \leq M$, let us pick $\epsilon_i > 0 : \forall y : |y - f_\mu^i(x)| < \epsilon_i, f_\mu^i(x_n) \in J_i$. In other words, pick some $\epsilon_i > 0$ s.t. $f_\mu^i(x_n)$ and $f_\mu^i(x)$ agree on the interval. Note that f_μ^i continuous everywhere $\forall i$. Thus, $\exists \delta_i > 0 : |y - x| < \delta_i \to |f_\mu^i(y) - f_\mu^i(x)| < \epsilon_i$. Take $\delta := \min\limits_{0 \leq i \leq M} \delta_i$. Because $x_n \to x$, $\exists N : n > N \to |x_n - x| < \delta$. Take this N. ■

Continuity of S^{-1}

Proof

We again use the sequential criterion of continuity. Consider arbitrary sequence $(y_n) \subset \Sigma_2$ with $y_n \to y \in \Sigma_2$. Define $x_i := S^{-1}(y_i) \in \Lambda \forall i$ and $x := S^{-1}(y)$; we want $x_i \to x$. Suppose not. Then there exists $\epsilon > 0$ and subsequence $(x_{n_k})_k \subset (x_n)_n$ s.t. $\forall k, |x_{n_k} - x| \geq \epsilon$. Observe that $\mu > 2 + \sqrt{5} \to |f'_\mu(x)| > 1 \forall x \in I_0 \cup I_1$. Then by MVT, $|f_\mu(x_{n_k}) - f_\mu(x)| > \lambda |x_{n_k} - x|$, for some $\lambda > 1$. This implies $|f^i_\mu(x_{n_k}) - f^i_\mu(x)| > \lambda^i |x_{n_k} - x| \geq \lambda^i \epsilon$, which is unbounded. Note that $y_n \to y$ means that $\forall M \in \mathbb{N}, \exists N : n > N \to \forall 0 \leq i \leq M$, $f^i_\mu(x_n)$ and $f^i_\mu(x)$ agree on whether they are in I_0 or I_1. But this is a contradiction, because for large enough M, $\forall N \exists n > N$ s.t. $|f^M_\mu(x_n) - f^M_\mu(x)| > len(I_0) = len(I_1)$. ■

Alternative Proof for S^{-1}

Proof

Note that Λ compact. By the Heine-Borel theorem, boundedness and totally boundedness are equivalent on $\mathbb{R}^n$. Obviously, Λ is closed and bounded. Recall that compact is equivalent to closed AND totally bounded.
Recall from topology that a continuous bijection with compact domain has continuous inverse. Done!

Bijection

Proof

Injectivity: Suppose $S(x) = S(y)$ for $x, y \in \Lambda$. Then $\forall n \in \mathbb{N}$, $f_\mu^n(x)$ and $f_\mu^n(y)$ agree on whether they are in I_0 or I_1. Suppose $x \neq y$. Then similar the previous proof, we have $|f_\mu^n(x_{n_k}) - f_\mu^n(x)| > \lambda^n |x_{n_k} - x| \geq \lambda^n \epsilon$. Contradiction.

Surjectivity: Define $I_{s_0 s_1 \cdots s_n} = \{x \in \Lambda : x \in I_{s_0}, f_\mu(x) \in I_{s_1}, \cdots, f_\mu^n(x) \in I_{s_n}\} = I_{s_0} \cap f_\mu^{-1}(I_{s_1 s_2 \cdots s_n})$. By induction, $I_{s_1 s_2 \cdots s_n}$ is closed and nonempty. By continuity of f, $f_\mu^{-1}(I_{s_1 s_2 \cdots s_n})$ is closed. Note that I_{s_0} closed, so because intersection of closed sets is closed, $I_{s_0 s_1 \cdots s_n}$ closed. Note that $f_\mu^{-1}(I_{s_1 s_2 \cdots s_n}) = I_{0 s_1 s_2 \cdots s_n} \cup I_{0 s_1 s_2 \cdots s_n}$, both of which are nonempty, so $I_{s_0 s_1 \cdots s_n}$ nonempty. Consider arbitrary $y \in \Sigma_2, S(y) = (s_0 s_1 \cdots)$, we have $\bigcap_{n=0}^{\infty} I_{s_0 \cdots s_n}$ is closed. Moreover, we know that it only contains 1 element: $f_\mu^{-1}(y)$. ∎

Topological Conjugacy

Theorem

$S \circ f = \sigma \circ S$.

Proof

We will show $f = S^{-1} \circ \sigma \circ S$. Suppose $x \in \Lambda$, $S(x) = (s_0 s_1 \cdots)$. Then $S^{-1} \circ \sigma \circ S(x) = S^{-1}(s_1 \cdots) = \bigcap_{n=1}^{\infty} I_{s_1 \cdots s_n}$, which is obviously equal to $f(x)$.

S as Topological Conjugate

- Due to the topological conjugacy between Λ and Σ_2 via S, all dynamics properties on the two spaces area equivalent.
- Specifically, we have periodic points are dense in Λ, and there exists dense orbit in Λ.
- Thus, we have f_μ chaotic in Λ for $\mu > 2 + \sqrt{5}$.
- This is actually true for $\mu > 4$, by first taking a homeomorphism between the logistic map and the tent map, then proving chaos for the tent map using a topological conjugacy between the tent map and Σ_2.

Questions?

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