Matroid theory and some applications

Ahmad

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Outline

Introduction to Matroid Theory

Matroids are powerful algebraic structures that generalise ideas of linear independence from linear algebra to arbitrary finite sets. They provide a useful framework for solving various problems, from network design to scheduling problems.

Key properties of rank and independence allow us to use greedy assumptions to find optimal answers for many optimization problems.

Independence System and Matroid Axioms

Independence System

Let N be a finite set and $\mathcal{I} \subseteq 2^N$. $(N, \mathcal{I})$ is an independence system if:

i) $\emptyset \in \mathcal{I}$

ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$

Matroid Axiom (Exchange)

An independence system $(N, \mathcal{I})$ is a *matroid* if:

iii) For all $A, B \in \mathcal{I}$ with $|B| > |A|$, there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$

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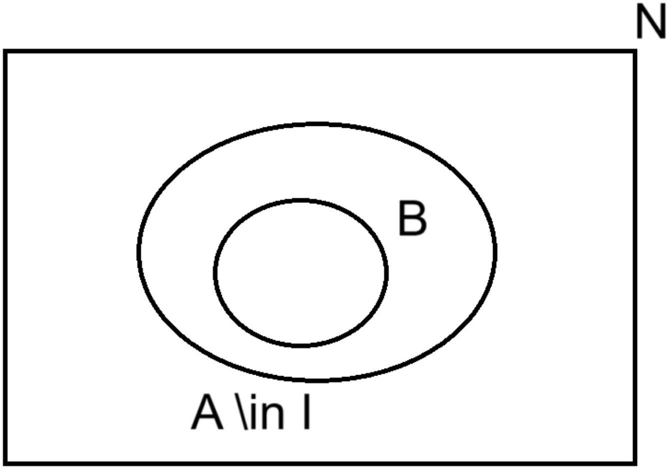
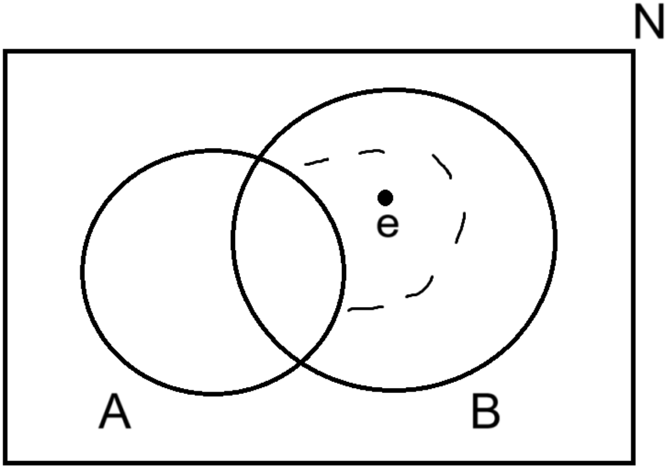
Hereditary Property

Hereditary property: Subsets of independent sets are independent.



Exchange Property

Exchange property: Between any two independent sets of different sizes, one can exchange elements to preserve independence.



Matrix Matroid

Definition

Given an $m \times n$ matrix M over a field, let $N = [n]$, and choose $\mathcal{I} = \{S \subseteq N : \{M_i : i \in S\}$ are linearly independent$\}$. Then $(N, \mathcal{I})$ is a matroid (matrix matroid).

Quick example of a matrix matroid

Let A be the following matrix with coefficients in $\mathbb{R}$:

$$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline & 1 & 0 & 0 & 1 & 1 \\ & 0 & 1 & 0 & 0 & 1 \end{array}$$

Then the family of independent sets $\mathcal{I}(M)$ contains, for example,

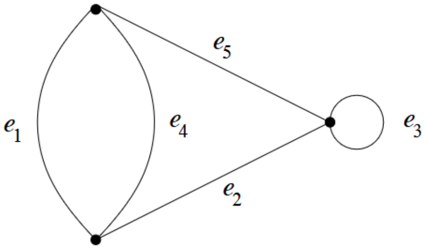
$$\big\{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}\big\} \subseteq \mathcal{I}(M).$$

Graphic Matroid

Definition

Given a graph $G = (V, E)$, let $N = E$, and $\mathcal{I} = \{F \subseteq E : (V, F) \text{ is a forest}\}$. Then $(N, \mathcal{I})$ is a matroid (graphic matroid). Recall that a forest is a graph with no cycles.

Quick example



It can be checked that $M(G)$ is isomorphic to $M(A)$ (under the bijection $e_i \mapsto i$).

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Rank function

The *rank* is analogous to bases in lin alg. The *rank* of a set $X \subseteq N$ is defined by

$$r_M(X) \;=\; \max\{\,|Y| : Y \subseteq X,\; Y \in \mathcal{I}\}.$$

Moreover, $r = r_M$ is the rank function of a matroid $(N, \mathcal{I})$, where

$$\mathcal{I} = \{\, I \subseteq N : r(I) = |I|\,\},$$

if and only if $r\colon 2^N \to \mathbb{Z}_{\geq 0}$ satisfies:

(R1) $0 \;\leq\; r(X) \;\leq\; |X|$ for all $X \subseteq N$,

(R2) $r(X) \;\leq\; r(Y)$ whenever $X \subseteq Y \subseteq N$,

(R3) $r(X \cup Y) + r(X \cap Y) \;\leq\; r(X) + r(Y)$ for all $X, Y \subseteq N$.

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Some examples of rank functions

1. **Matrix Matroid.** Let M be an $m \times n$ matrix over a field, and let $(N = [n], \mathcal{I})$ be its associated matrix matroid, where $\mathcal{I} = \{S \subseteq N : \{M_i : i \in S\}$ are linearly independent$\}$. Then for any $A \subseteq N$,

$$r(A) \;=\; \max\big\{|S| : S \subseteq A,\ S \in \mathcal{I}\big\} \;=\; \mathrm{rank}\big([M_i : i \in A]\big).$$

2. **Graphic Matroid.** Let $G = (V, E)$ be a graph, and let $(N = E, \mathcal{I})$ be the graphic matroid, where $\mathcal{I} = \{F \subseteq E : (V, F)$ is a forest$\}$. Then for any $A \subseteq E$,

$$r(A) \;=\; |V| \;-\; \big(\text{number of connected components of } (V, A)\big).$$

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Greedy Algorithm on Matroids

Matroids allow us to use greedy assumptions to find exact optimal solutions to specific optimization problems

- For any weight function $w : N \to \mathbb{R}$, the greedy algorithm
  1. Start with $X \leftarrow \emptyset$.
  2. While there exists $e \notin X$ such that $X \cup \{e\} \in \mathcal{I}$, pick the remaining element of maximum $w(e)$.
- The greedy choice yields a maximum-weight independent set *iff* $(N, \mathcal{I})$ is a matroid.

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Transversal Matroid and Assignment

Transversal Matroid

Let tasks $T = \{t_1, \ldots, t_n\}$ and agents $A = \{e_1, \ldots, e_m\}$ with edges whenever agent can perform task. Partial transversals correspond to matchings in the bipartite graph. The family of partial transversals is the set of independent sets of a matroid.

Assignment Problem

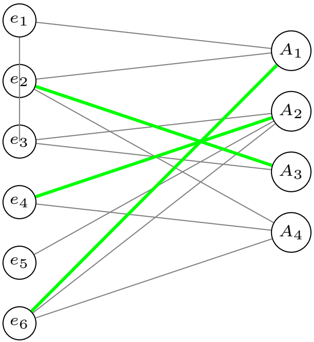
Given priorities $w(t)$ on tasks, greedy on the transversal matroid selects tasks in decreasing w to maximize total priority.

Partial Transversal Example

Let $E = \{e_1, \ldots, e_6\}$ and $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ with

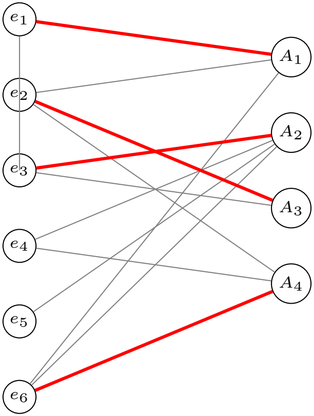
$$A_1 = \{e_1, e_2, e_6\}, \quad A_2 = \{e_3, e_4, e_5, e_6\}, \quad A_3 = \{e_2, e_3\}, \quad A_4 = \{e_2, e_4, e_6\}.$$

Then $X = \{e_6, e_4, e_2\}$ is a *partial* transversal of $\mathcal{A}$, since it matches into $\{A_1, A_2, A_3\}$.



Full Transversal Example

With the same ground sets, $\{e_1, e_3, e_2, e_6\}$ is a *transversal* of $\mathcal{A}$.



Max Weight Independent Set Problem

Matroid Optimization

Given a matroid $(N, \mathcal{I})$ and weight $w : N \to \mathbb{R}$, find $\max\{\sum_{e \in I} w(e) : I \in \mathcal{I}\}$.

Special cases:

- Max weight forest (graphic matroid) — corresponds to min spanning tree when weights are negated.
- Max weight independent columns (matrix matroid).

Conclusion

- Matroids capture the essence of independence across various domains.
- Greedy assumptions work optimally if you're dealing with matroid formulations.
- Applications cover many realms in optimization be it assignment problems or matchings.

References

- Theory of matroids and applications : I from J.L. Ramirez Alfonsin here
- Lecture 13 on Matroids from Karthik Chandrasekaran's IE 511 notes here

I left Cambridge in 1941 with the idea that graph theory could be reduced to abstract algebra but that it might not be the conventional kind of algebra

— W.T Tutte (1940-something)