World's Simplest Poker

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Disclaimer

- This presentation is based on an IML (Illinois Mathematics Lab) project I am working on, “The Mathematics of Poker-Like Games", under Professor Hildebrand.
- The IML project is inspired by “World’s Simplest Poker" (2012) by Professor David McAdams of Duke University.
- As we are still researching, some ideas in this presentation remain open/not fully explored.
- I might do a (short) presentation later in the semester exploring these ideas.
- The point of this talk is not to teach you the best poker strategy or make you a better poker player.

Outline

Section 1

Introduction

World's Simplest Poker

- *Poker* is a game of betting, luck and strategy.
- We consider a (very) simplified version of poker, in which there are two players A and B, and n cards labeled 1 to n.
- Each player is dealt a random card without replacement.
- The initial ante to play is \$1, then players can choose to *bet* an additional \$1 or *fold*, independently of the other player's choice.
- If one player bets, and the other folds, that player wins \$1.
- If both players fold, the player with the higher card wins \$1.
- If both players bet, the player with the higher card wins \$2.
- Note that "fold" doesn't mean losing the hand right away!

Example Game

- Let $n = 3$. (This is often referred to as *A-K-Q Poker*), and that the other player bets if and only if they get card 2 or 3.
- Suppose you get a 1. Should you bet?
  Answer: No
- Suppose you get a 3. Should you bet?
  Answer: Yes
- Suppose you get a 2. Should you bet?
  Answer: No
- $\mathbb{E}$ from betting: $\mathbb{E}_2 = 1 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = -\frac{1}{2}$.
- $\mathbb{E}$ from not betting: $\mathbb{E}_2 = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$.
- Note that we consider a player's strategy *independently* of the other player's bet. For all we know, the could be bluffing!

More Examples

- Define a player's *betting set* $S_A, S_B \subseteq \{1, 2, 3, \ldots, n\}$ (S_A for player A, S_B for player B) such that a player bets on card c if and only if $c \in S_A, S_B$.
- Suppose player A has a betting set $S_A = \{\}$. What is player B's best response?
  Answer: $S_B = \{1, 2, 3\}$. Player B will always win!

More Examples

- Suppose player A has a betting set $S_A = \{1, 2, 3\}$. What is player B's best response?
  Answer:
  - You should always bet on 3, never on 1.
  - If you bet on 2, your expected payout is $\mathbb{E}_2 = 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = 0$.
  - If you fold on 2, your expected payout is $\mathbb{E}_2 = -\frac{1}{2} - \frac{1}{2} = -1$, since A always bets.
  - Thus, $S_A = \{2, 3\}$.

Game Theory Fundamentals

- A *zero-sum* game has a expected payout sum over all players of 0. Poker is, of course, a zero-sum game.
- We say a player has a *dominant strategy* if it is the best strategy for the player, regardless of what the other player chooses to do.
- We call a strategy *deterministic* if it always produces the same outcome for a given input, without any randomness or uncertainty (We have only considered deterministic strategies so far).
- A *Nash Equilibrium* occurs when neither player can gain a higher payout by changing their current strategy.

Payout Matrix for $n = 3$

- We can compute all of the results for $n = 3$, (either by hand or python code simulating all the games in $\mathcal{O}(2^n \cdot 2^n \cdot n^2) = \mathcal{O}(4^n n^2)$ time)
- What would dominant strategies or Nash Equilibria look like in this table?

$S_B \backslash S_A$	{}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{}	0	4	2	0	6	4	2	6
{1}	-4	0	1	-1	5	3	4	8
{2}	-2	-1	0	1	1	2	3	4
{3}	0	1	-1	0	0	1	-1	0
{1,2}	-6	-5	-1	0	0	1	5	6
{1,3}	-4	-3	-2	-1	-1	0	1	2
{2,3}	-2	-4	-3	1	-5	-1	0	-2
{1,2,3}	-6	-8	-4	0	-6	-2	2	0

Payout Matrix for $n = 3$

- That's right, there are none! We will continue to explore this idea later...

Section 2

The Cutoff Strategy

The Cutoff Strategy

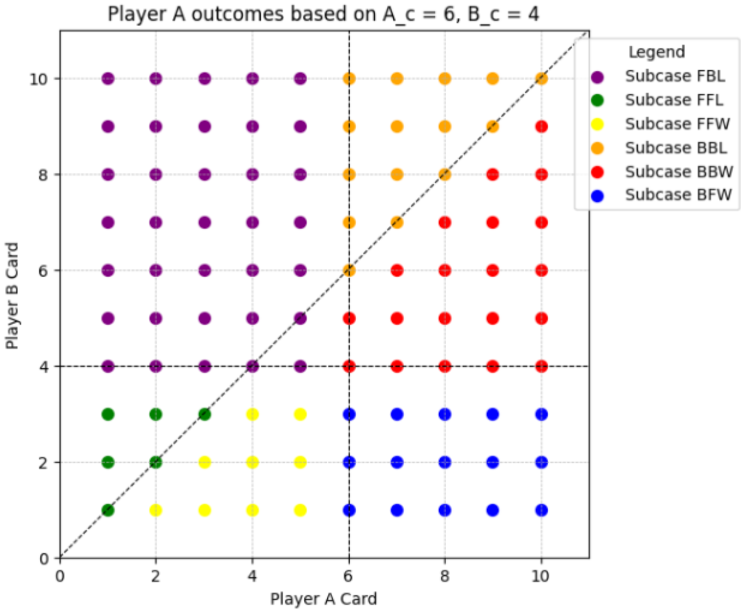
- A common deterministic strategy players use is called the *cutoff* strategy, where a player bets if and only the card c_i they receive is greater than or equal to a certain cutoff value $1 \leq A_c, B_c \leq n + 1$.
- What would be the betting set of a player using a cutoff strategy of $A_c = 2$? (Assume $n = 3$).
  Answer: $S_A = \{2, 3\}$.

Analysis of The Cutoff Strategy

- Using this subset of all possible deterministic strategies, we can find the expected payout of each player given their cutoff values A_c, B_c and n.
- WLOG, assume that $A_c > B_c$. We find a formula for player A's expected payout $\mathbb{E}_A$ and use the zero-sum game fact as $\mathbb{E}_A = -\mathbb{E}_B$.
- What about when $A_c = B_c$? What is $\mathbb{E}_A$ in this case?
  Answer: By symmetry, $\mathbb{E}_A = \mathbb{E}_B = 0$.

Analysis of The Cutoff Strategy

- We end up with 6 cases to consider:
    1. FFL
    2. FFW
    3. BFW
    4. FBL
    5. BBW
    6. BBL
- The first letter is player A's action (fold or bet), the second letter is player B's action (fold or bet), and the last letter is the outcome for the case for player A (win or lose).
- What happened to the BFL and FBW cases? *Answer:* If player A bets and player B folds, player A always wins.


Player A outcomes based on A_c = 6, B_c = 4
Legend
Subcase FBL
Subcase FFL
Subcase FFW
Subcase BBL
Subcase BBW
Subcase BFW
Player B Card
Player A Card
0
2
4
6
8
10
0
2
4
6
8
10

Analysis of the Cutoff Strategy

- The total expected payout is then:

$$\mathbb{E}_A = \sum_{i=1}^{6} \mathbb{E}_{\text{case } \#\text{i}} \cdot P_{\text{case } \#\text{i}},$$

  where $\mathbb{E}_{\text{case } \#\text{i}}$ and $P_{\text{case } \#\text{i}}$ are the expected profit and probability given that x and y satisfy the relations for case $\#i$.

- In this presentation, I will only go through one case, as the derivation is similar for the other five.

The FFL Case

- This case arises in the scenario that neither player bets and Player A loses the showdown.
- First, we compute the probability that this case occurs.
- Let c_A and c_B be the card players A and B recieve, respectively. There are $n \cdot (n-1)$ ways to choose c_A and c_B.
- There are $(B_c - 1)$ ways to pick $c_B < B_c$.
- Since player A loses this hand, we need $c_A < c_B < B_c$ as well, and there are $(B_c - 2)$ cards left satisfying $c_A < B_c$.
- By symmetry, exactly $1/2$ of these cases satisfy $c_A < c_B$.
- Thus, we get

$$P_1 = \frac{(B_c - 1)(B_c - 2)}{2n(n-1)}.$$

The FFL Case

- Note that $\mathbb{E}_{\text{case }\#1} = -1$, as player A loses the showdown and neither player bets. Thus, we have

$$\mathbb{E}_{\text{case }\#1} \cdot P_{\text{case }\#1} = -1 \cdot \frac{(B_c - 1)(B_c - 2)}{2n(n-1)} = -\frac{(B_c - 1)(B_c - 2)}{2n(n-1)}.$$

The other cases

- Doing the same for the other 5 cases, we find the following results:

Case	Player A Profit	Probability
BBW	+2	$\frac{(n+1-A_c)(n+A_c-2B_c)}{2n(n-1)}$
BBL	-2	$\frac{(n+1-A_c)(n-A_c)}{2n(n-1)}$
FFW	+1	$\frac{(B_c-1)(2A_c-B_c-2)}{2n(n-1)}$
FFL	-1	$\frac{(B_c-1)(B_c-2)}{2n(n-1)}$
BFW	+1	$\frac{(n-A_c+1)(B_c-1)}{n(n-1)}$
FBL	-1	$\frac{(A_c-B_c)(n-B_c)+(B_c-1)(n-B_c+1)}{n(n-1)}$

Putting it all together

- Going through the messy algebra, we eventually get that

$$\mathbb{E}_A = \sum_{i=1}^{6} \mathbb{E}_{\text{case \#i}} \cdot P_{\text{case \#i}} \implies$$

$$\mathbb{E}_A = \frac{3A_cB_c - B_c^2 - 2A_c^2 + 2A_c - 2B_c + nA_c - nB_c}{n(n-1)}, \{A_c > B_c\}.$$

- Since we have a zero-sum game, we have

$$\mathbb{E}_B = -\mathbb{E}_A = \frac{-3A_cB_c + B_c^2 + 2A_c^2 - 2A_c + 2B_c - nA_c + nB_c}{n(n-1)}, \{A_c > B_c\}.$$

Putting it all Together

- Since the game is symmetric between the two players, we can swap all of the indices A and B to derive the case when $B_c > A_c$.

$$\mathbb{E}_A = \frac{-3A_cB_c + A_c^2 + 2B_c^2 - 2B_c + 2A_c - nB_c + nA_c}{n(n-1)}, \{B_c > A_c\}.$$

- Thus, we have found a (pretty ugly) formula for player A's expected payout given both cutoffs A_c, B_c, and n.

MatPlotlib Python Plot

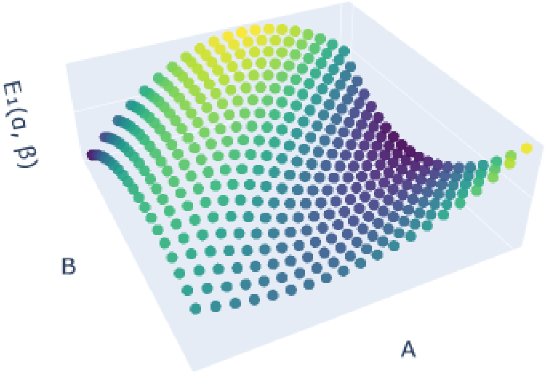
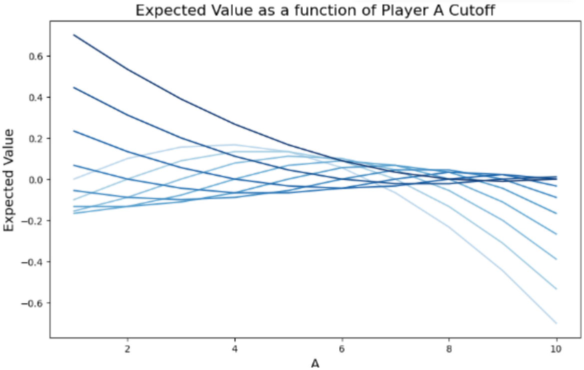


Figure: Expected Payoff of Player 1 in terms of cutoffs A and B

Level Curves

- To better visualize this 3D space, we can plot *level curves*, where we hold B_c constant and plot $\mathbb{E}$ vs. A_c:



- Darker curves indicate larger values of B_c.

Finding Player 1's Maximum Payout

- We now maximize $\mathbb{E}_A$ over A_c for a fixed B_c and n.
- Suppose that player A adopts a strategy with $A_c > B_c$.
- Then player 1's expected payout is given by

$$\mathbb{E}_A = \frac{3A_cB_c - B_c^2 - 2A_c^2 + 2A_c - 2B_c + nA_c - nB_c}{n(n-1)}.$$

- We re-write this as a quadratic in A_c :

$$\mathbb{E}_A = \frac{-2A_c^2 + (2 + n + 3B_c)A_c - (B_c^2 + nB_c + 2B_c)}{n(n-1)}.$$

- Then, we take the derivative with respect to A_c (B_c is fixed)

$$\frac{\partial}{\partial A_c}\mathbb{E}_A = \frac{-4A_c + (2 + n + 3B_c)}{n(n-1)}$$

Finding Player 1's Maximum Payout ($A_c > B_c$)

- This function has a maximum at

$$\frac{\partial}{\partial A_c}(\mathbb{E}_A) = 0 \implies \frac{-4A_c + (2 + n + 3B_c)}{n(n-1)} = 0 \implies A_c^* = \frac{2 + n + 3B_c}{4}.$$

- We can verify that A_c^* satisfies $B_c < A_c^* \leq n + 1$, for all $1 \leq B_n \leq n + 1$, so this choice of $A_c = A_c^*$ is valid.
- Thus, an optimal choice of A_c for strategies satisfying $A_c > B_c$ is

$$A_c^* = \left\lfloor \frac{2 + n + 3B_c}{4} \right\rceil,$$

where $\lfloor x \rceil$ is the closest integer function.

Finding Player 1's Maximum Payout $(A_c < B_c)$

- We have
$$\mathbb{E}_A = \frac{A_c^2 + (n+2-3B_c)A_c - (nB_c + 2B_c - 2B_c^2)}{n(n-1)}.$$
- Since this parabola opens up, we consider the endpoints. $A_C = \{1, B_1 - 1\}$.
- $A_c = 1$: We have
$$\mathbb{E}_A = \frac{1^2 + (n+2-3B_c)\cdot 1 - (nB_c + 2B_c - 2B_c^2)}{n(n-1)} \implies$$
$$\mathbb{E}_A = \frac{3 + n - 5B_c - nB_c + 2B_c^2}{n(n-1)}.$$

Finding Player 1's Maximum Payout $(A_c > B_c)$

- $A_c = B_c - 1$: We have

$$\mathbb{E}_A = \frac{(B_c - 1)^2 + (n + 2 - 3B_c)(B_c - 1) - (nB_c + 2B_c - 2B_c^2)}{n(n-1)} \implies$$

$$\mathbb{E}_A = \frac{B_c - n - 1}{n(n-1)}.$$

- But since $B_c \leq n + 1$, we have $\mathbb{E}_A \leq 0$ in this case, so player A would never choose this strategy.

Finding Player 1's Maximum Payout

- Thus, given B_c, the optimal value of A_c is either 1 or $\left\lfloor \frac{2+n+3B_c}{4} \right\rceil$, depending on which gives a larger value for $\mathbb{E}_A$.
- In general, player A should use $A_c = 1$ while B_c is high, then switch to $\left\lfloor \frac{2+n+3B_c}{4} \right\rceil$ when player B's cutoff gets low enough.
- Finding the exact value for when to switch is quite annoying due to the closest integer function.
- Instead, we find a useful approximation that lets us drop the closest integer.

The Continuous Analog

- As n grows large, we can ignore the discrete differences in cards, the without replacement condition, (and drop the closest integer function).
- Recall that n is the number of potential *hands*, so for a 52 card deck, this could be as large as $\binom{52}{2} = 1326$.
- Then, we can reframe our problem as choosing cutoffs α_c and β_c such that $\alpha_c, \beta_c \in \mathbb{R}$ and $\alpha_c, \beta_c \in [0, 1]$, and frame the cards chosen as random real numbers in $[0, 1]$.
- We can derive the continuous versions of our formulas by approximating $B_c \approx n\beta_c$, $A_c \approx n\alpha_c$, and taking the limit as $n \to \infty$.

The Continuous Analog - Expected Profit

- Substituting $A_c = n\alpha_c$ and $B_c = n\beta_c$ into the formula for $\mathbb{E}_A$ (assuming $\beta_c > \alpha_c$), we obtain

$$\mathbb{E}_A = \frac{2\beta_c^2 n^2 - (n + 2 + 3n\alpha_c)n\beta_c + (n^2\alpha_c^2 + (n+2)n\alpha_c)}{n(n-1)}$$

- Taking the limit as $n \to \infty$, only the quadratic terms survive.

$$\lim_{n\to\infty} \mathbb{E}_A = \lim_{n\to\infty} \frac{n^2(2\beta_c^2 - \beta_c - 3\alpha_c\beta_c + \alpha_c^2 + \alpha_c)}{n(n-1)} = 2\beta_c^2 - \beta_c - 3\alpha_c\beta_c + \alpha_c^2 + \alpha_c.$$

- We can similarly derive $\mathbb{E}_A$ in the case $\alpha_c > \beta_c$:

$$\mathbb{E}_A = -2\alpha_c^2 - \beta_c^2 + 3\alpha_c\beta_c + \alpha_c - \beta_c$$

The Continuous Analog - Maximization

- We can compute the α_c^* (best cutoff values for player A as $n \to \infty$), rescaling to $[0, 1]$, using our formulas from before.

$$A_c^* = 1 \implies \alpha_c^* = 0, \{\alpha_c < \beta_c\}$$

$$A_c^* = \left\lfloor \frac{2 + n + 3B_c}{4} \right\rceil \implies \alpha_c^* = \frac{3\beta + 1}{4}, \{\alpha_c > \beta_c\}$$

- Plugging these back to $\mathbb{E}_A$:

$$\mathbb{E}_A(0) = 2\beta_c^2 - \beta_c, \{\alpha_c < \beta_c\}.$$

$$\mathbb{E}_A\left(\frac{3\beta + 1}{4}\right) = \frac{(\beta_c - 1)^2}{8}, \{\alpha_c > \beta_c\}.$$

The Continuous Analog - Maximization

- Now, we find for which β_c we should choose $\alpha^* = 0$ or $\alpha^* = \frac{3\beta+1}{4}$.

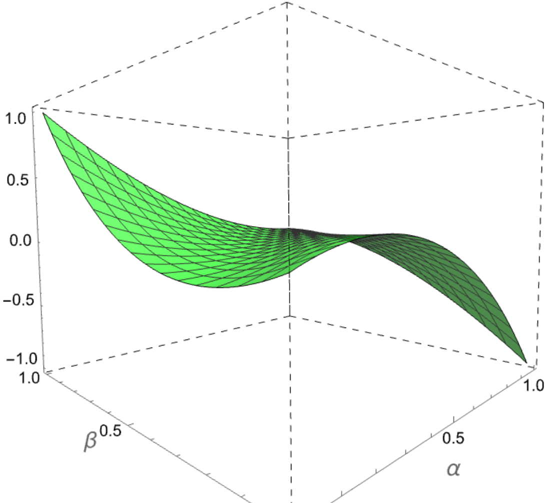
$$\mathbb{E}_A(0) > \mathbb{E}_A\left(\frac{3\beta+1}{4}\right) \iff 2\beta_c^2 - \beta_c > \frac{(\beta_c - 1)^2}{8}$$

$$\iff 15\beta_c^2 - 6\beta_c - 1 > 0 \iff \beta_c > \frac{3+2\sqrt{6}}{15} \approx 0.5266.$$

- Thus, player A should play with a cutoff strategy $\alpha = 0$ iff $\beta_c > \frac{3+2\sqrt{6}}{15}$, otherwise player A should use a cutoff of $\frac{3\beta+1}{4}$.
- If player B chooses their best strategy with $\beta_c^* = \frac{3+2\sqrt{6}}{15}$, then player A can achieve a maximum expected payout of

$$\mathbb{E}_A = \frac{(\beta_c^* - 1)^2}{8} = \frac{7-2\sqrt{6}}{75} \approx 0.028.$$

The Continuous Analog - Graph



Section 3

General Strategies

Dominance of Deterministic Strategies

- Up until now, we have focused on analyzing the cutoff strategy and the expected payouts gained from it.
- But is the cutoff strategy a dominant strategy?
- That is, no matter what player B does, can player A achieve their maximum possible expected profit by using a cutoff strategy?

Dominance of Deterministic Strategies

- Consider $S_B = \{3\}$. (Recall that S_B is the *betting set* of player B).
- What is player A's best response?

$S_B \backslash S_A$	{}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{}	0	4	2	0	6	4	2	6
{1}	-4	0	1	-1	5	3	4	8
{2}	-2	-1	0	1	1	2	3	4
{3}	0	1	-1	0	0	1	-1	0
{1,2}	-6	-5	-1	0	0	1	5	6
{1,3}	-4	-3	-2	-1	-1	0	1	2
{2,3}	-2	-4	-3	1	-5	-1	0	-2
{1,2,3}	-6	-8	-4	0	-6	-2	2	0

Table: Payoff matrix for A and B strategies

Dominance of Deterministic Strategies

- Answer: $S_A = \{1\}$ or $\{1, 3\}$... hey wait, neither of those are cutoff strategies!
- It follows that only playing cutoff strategies is not optimal, and there are situations where other strategies are better.
- However, it can be shown that a cutoff strategy is optimal for (nearly all) possible bluffing sets of the other player.
- Thus, the cutoff strategy is still “generally" good.

Repeated Iterations of Optimal Strategies

- Consider this scenario:
  - Player A picks a strategy S_A
  - Player B picks their optimal response S_B
  - Player A then changes their strategy to pick their best possible response given player Bs current strategy.
  - Then player B changes their strategy, and the cycle repeats.
- This leads to a sequence of iterated strategies, which we will call $s_1, s_2, s_3, \ldots, s_T$.
- Note that this sequence must eventually be periodic, as there are only finitely many possible values of s_i, and s_{i+1} only depends on the value of s_i.
- Let's call the period of such a “strategy cycle" T.

Repeated Iterations of Optimal Strategies

- Question: What other term refers to a strategy cycle of length 1 (i.e. $T = 1$)? *Answer:* A Nash Equilibrium!

Repeated Iterations of Optimal Strategies

- For example, suppose $n = 3$ and initially player B bets on $S_B = \{2, 3\}$.

$S_B \backslash S_A$	{}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{}	0	4	2	0	6	4	2	6
{1}	-4	0	1	-1	5	3	3	8
{2}	-2	-1	0	1	1	2	3	4
{3}	0	1	-1	0	1	-1	-1	0
{1,2}	-6	-5	-1	0	0	1	5	6
{1,3}	-4	-3	-2	-1	-1	0	1	2
{2,3}	-2	-4	-3	1	-5	-1	0	-2
{1,2,3}	-6	-8	-4	0	-6	-2	2	0

- What is player $A's$ best response? Answer: $S_A = \{3\}$.

Repeated Iterations of Optimal Strategies

$S_B \backslash S_A$	{}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{}	0	4	2	0	6	4	2	6
{1}	-4	0	1	-1	5	3	3	8
{2}	-2	-1	0	1	1	2	3	4
{3}	0	1	-1	0	1	-1	-1	0
{1,2}	-6	-5	-1	0	0	1	5	6
{1,3}	-4	-3	-2	-1	-1	0	1	2
{2,3}	-2	-4	-3	1	-5	-1	0	-2
{1,2,3}	-6	-8	-4	0	-6	-2	2	0

- Now, what would player B do to maximize their payout?
- Player B would bet on $\{1\}$ or $\{1,3\}$. (It turns out it doesn't matter which one is picked, so we assume $S_B = \{1,3\}$ in this case.)

Repeated Iterations of Optimal Strategies

- We can continue this process until we arrive back at where we started.

$S_B \backslash S_A$	{}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
{}	0	4	2	0	6	4	2	6
{1}	-4	0	1	-1	5	3	3	8
{2}	-2	-1	0	1	1	2	3	4
{3}	0	1	-1	0	1	-1	-1	0
{1,2}	-6	-5	-1	0	0	1	5	6
{1,3}	-4	-3	-2	-1	-1	0	1	2
{2,3}	-2	-4	-3	1	-5	-1	0	-2
{1,2,3}	-6	-8	-4	0	-6	-2	2	0

- The cycle now repeats. Thus, the period is $T = 4$.

Repeated Iterations of Optimal Strategies

- We know that there are cycles of length 4 for all $n \leq 10$, as well as cycles of length 10 for $n \geq 4$.
- However, we have yet to prove that these are the only ones yet.

Bluffing

- A common *non-deterministic* strategy is *bluffing*.
- A player with card c, betting cutoff A, and bluffing probability p will bet with probability p if $c < A$, and always bet with $c \geq A$.
- We haven't looked to much into this, but it has been shown that, in the continuous case, if both players have the same betting cutoff and bluffing probability, there is a nash equilibrium at

$$(p_A, A_c) = (p_B, B_C) = \left(\frac{1}{3}, \frac{1}{2}\right),$$

due to David McAdams (2012).

Future Work

- There are several questions we have yet to answer, here are some of the most pressing mysteries.
- Given that player B chooses a betting set S_B, what is player A's best possible response, in terms of S_B?
- Does there exist a nash equilbrium in the discrete case considering deterministic strategies?
- Are there any nash equilbria or dominant strategies when bluffing is introduced?
- We hope to find some answers to these questions as we continue our research.

Python Code

- Link to Colab Notebook

Questions?

Brainteaser

- Let S be the set of all distinct triangles (i.e, no two triangles in S are congruent) that have perimeter 2025 and integer degree angle measures. A random triangle is chosen from S. What is the probability that the selected triangle is obtuse?

Bibliography I

- McAdams, David. "World's Simplest Poker" *Duke University*, 2013, https://cheaptalk.org/wp-content/uploads/2012/11/worlds-simplest-poker.pdf