World's Simplest Poker (Bluffing)

Mihir Tandon

Disclaimer

- This presentation is based on an IML (Illinois Mathematics Lab) project I am working on, "The Mathematics of Poker-Like Games", under Professor Hildebrand.
- The IML project is inspired by "World's Simplest Poker" (2012) by Professor David McAdams of Duke University.
- This is a short follow-up to the first one, where we will be focusing on mixed strategies.
- The point of this talk is not to teach you the best poker strategy or make you a better poker player.

Outline

Recap

Mixed Strategies

Section 1

Recap

World's Simplest Poker

- *Poker* is a game of betting, luck and strategy.
- We consider a (very) simplified version of poker, in which there are two players A and B, and n cards labeled 1 to n.
- Each player is dealt a random card without replacement.
- The initial ante to play is \$1, then players can choose to *bet* an additional \$1 or *fold*, independently of the other player's choice.
- If one player bets, and the other folds, that player gets a profit of \$1.
- If both players fold, the player with the higher card gets a profit of \$1.
- If both players bet, the player with the higher card gets a profit of \$2.

Game Theory Fundamentals

- A *zero-sum* game has a expected payout sum over all players of 0. Poker is, of course, a zero-sum game.
- We call a strategy *deterministic* if it always produces the same outcome for a given input, without any randomness or uncertainty.
- A *Nash Equilibrium* occurs when neither player can gain a higher payout by changing their current strategy.
  - Result from last time: No nash equilibria over deterministic strategies for $n \geq 3$. (at least for $n \leq 10$, but almost certainly true in general)

Section 2

Mixed Strategies

Introduction

- Last time, we only considered deterministic strategies (i.e. having no randomness).
- A *mixed strategy* is when a player assigns probabilities (randomness) to different pure strategies.
- We will consider the most general strategy space in which each player bets on a card c_i with some probability $0 \le p_i \le 1$.
- We can then represent a player's strategy as a probability vector

$$\mathcal{P}_A = \vec{p_i} = (p_1, p_2, p_2, \ldots, p_n)$$

for player A, and

$$\mathcal{P}_B = \vec{q_i} = (q_1, q_2, q_3, \ldots, q_n)$$

for player B.

Analysis for $n = 3$

- This ends up being hard to analyze for large n, so let's limit ourselves to the $n = 3$ case for now.
- Let
$$\mathcal{P}_A = (p_1, p_2, p_3), \mathcal{P}_B = (q_1, q_2, q_3),$$
be player A's and B's strategies, respectively.
- Let x and y be player A's and B's cards, respectively.
- Note that if we fix x and y, the expected profit $\mathbb{E}_A$ only depends on p_x, q_y, and the condition $x < y$.
- We can then consider the 4 possible "showdown" scenarios: Bet/Bet, Bet/Fold, Fold/Bet, Fold/Fold (BB,BF,FB,FF).

$x < y$ case

- We have the following contributions to player A's expected profit: (in terms of p_x and q_y)
  - BB: $-2 \cdot p_x q_y = -2p_x q_y$
  - BF: $1 \cdot p_x(1 - q_y) = p_x(1 - q_y)$
  - FB: $-1 \cdot (1 - p_x)q_y = -(1 - p_x)q_y$
  - FF: $-1 \cdot (1 - p_x)(1 - q_y) = -(1 - p_x)(1 - q_y)$.
- Combining these gives:

$$\mathbb{E}_{x<y} = -2p_x \cdot q_y + p_x(1-q_y) - (1-p_x)q_y - (1-p_x)(1-q_y) = -1 + 2p_x - 3p_x q_y.$$

$x > y$ case

- Same as the previous case but the signs for the symmetric (FF,BB) cases are flipped:
  - BB: $2p_x q_y$
  - BF: $p_x(1 - p_y)$
  - FB: $-p_y(1 - p_x)$
  - FF: $(1 - p_x)(1 - q_y)$.
- Combining these gives:

$$\mathbb{E}_{x>y} = 2p_x \cdot q_y + p_x(1-q_y) - (1-p_x)q_y + (1-p_x)(1-q_y) = 1 - 2q_y + 3p_x q_y.$$

Expected General Profit

Given these formulas, we can list out the $3 \cdot 2 = 6$ cases and find a general form for player A's expected profit:

$$\mathbb{E}_A = \frac{\mathbb{E}_{1<2} + \mathbb{E}_{1<3} + \mathbb{E}_{2<3} + \mathbb{E}_{3>2} + \mathbb{E}_{3>1} + \mathbb{E}_{2>1}}{6} =$$

$$\mathbb{E}_A = \frac{1}{6}((-1 + 2p_1 - 3p_1q_2) + (-1 + 2p_1 - 3p_1q_3) + (-1 + 2p_2 - 3p_2q_3) + (1 - 2q_1 + 3p_2q_1) + (1 - 2q_1 + 3p_3q_1) + (1 - 2q_2 + 3p_3q_2)) \implies$$

$$\mathbb{E}_A = \frac{(4 - 3q_3 - 3q_2)p_1 + (2 + 3q_1 - 3q_3)p_2 + (3q_1 + 3q_2)p_3 - 4q_1 - 2q_2}{6}.$$

Finding Best Strategies

$$\mathbb{E}_A = \frac{(4 - 3q_3 - 3q_2)p_1 + (2 + 3q_1 - 3q_3)p_2 + (3q_1 + 3q_2)p_3 - 4q_1 - 2q_2}{6}.$$

- Let's work from the perspective of player A.
- Since $3q_1 + 3q_2 \geq 0$, it is optimal for $p_3 = 1$, as well as $q_3 = 1$. (This should also be clear intuitively, you should always bet on the highest card).
- Substituting $p_3 = q_3 = 1$ gives

$$\mathbb{E}_A = \frac{(1 - 3q_2)p_1 + (3q_1 - 1)p_2 - q_1 + q_2}{6}.$$

- Depending on q_1 and q_2, we can find player A's best strategy.

Exact Optimal Strategy

- $q_2 \geq \frac{1}{3}$ and $q_1 \leq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (0, 0, 1)$, and
$$\mathbb{E}_A = \frac{q_2 - q_1}{6}.$$
- $q_2 \geq \frac{1}{3}$ and $q_1 \geq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (0, 1, 1)$, and
$$\mathbb{E}_A = \frac{2q_1 + q_2 - 1}{6}.$$
- $q_2 \leq \frac{1}{3}$ and $q_1 \leq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (1, 0, 1)$, and
$$\mathbb{E}_A = \frac{-q_1 - 2q_2 + 1}{6}.$$
- $q_2 \leq \frac{1}{3}$ and $q_1 \geq \frac{1}{3}$, which gives $(p_1, p_2, p_3) = (1, 1, 1)$, and
$$\mathbb{E}_A = \frac{2q_1 - 2q_2}{6}.$$

3D Plot

Desmos Link

Nash Equilibrium - Indifference

- At first glance, it might seem like it is always optimal to play with a deterministic strategy.
- However, player B can select q_1, q_2 such that player A is *indifferent* about their strategies (betting or folding on cards 1 and 2).
  - If Player A was not indifferent, they would just pick the strategy that gives them the highest expected payout.
- It follows that the coefficients in our formula

$$\mathbb{E}_A = \frac{(1 - 3q_2)p_1 + (3q_1 - 1)p_2 - q_1 + q_2}{6}$$

for p_1 and p_2 must be equal to 0. Thus $q_1 = 1/3$ and $q_2 = 1/3$.

Nash Equilibrium

- Note that our formula reduces to

$$\mathbb{E}_A = \frac{(1-3q_2)p_1 + (3q_1-1)p_2 - q_1 + q_2}{6} =$$

$$\frac{(1-1)p_1 + (1-1)p_2 - 1/3 + 1/3}{6} = 0.$$

- Thus, it follows that player A gets a payout of 0 no matter what they do, and thus is indifferent about their strategy.
- By symmetry, we find that $p_1 = p_2 = 1/3$, so a nash equilibrium exists at

$$(p_1, p_2, p_3) = (q_1, q_2, q_3) = (1/3, 1/3, 1)$$

in the $n = 3$ case.

Extension for higher $n \geq 4$

- The computation gets difficult for large n, since there are $n \cdot (n-1)$ cases to go through to even find the formula $\mathbb{E}_A$.
- However, going through the cases, we found the (unique) nash equilbrium for $n = 4$ occurs at the strategy $(0, 2/3, 1, 1)$ for both players.
- *Nash's Theorem* states that any game with a finite number of players and a finite number of strategies must have at least one mixed strategy equilibrium.

Future Work

- Generalizing Nash Equilbria for large n: Is there any pattern/way to generalize the nash equilbrium formula for any n?
- Generalizing betting values: Suppose we vary the betting size b into the pot. How do the Nash Equilbria change?

Questions?

Brainteaser

- Find all ordered pairs of primes (p, q) such that

$$p^2 - 43p + 110 = q^2 + 12q.$$

Bibliography I

- McAdams, David. "World's Simplest Poker" *Duke University*, 2013, https://cheaptalk.org/wp-content/uploads/2012/11/worlds-simplest-poker.pdf